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STABLE TOPOLOGICAL CYCLIC HOMOLOGY IS TOPOLOGICAL HOCHSCHILD HOMOLOGY

By LARS HESSELHOLT

1. INTRODUCTION

1.1. Topological cyclic homology is the codomain of the cyclotomic trace from algebraic K-theory

$$\operatorname{trc}: K(L) \to \mathrm{TC}(L).$$

It was defined in [2] but for our purpose the exposition in [6] is more convenient. The cyclotomic trace is conjectured to induce a homotopy equivalence after p-completion for a certain class of rings including the rings of algebraic integers in local fields of possitive residue characteristic p. We refer to [11] for a detailed discussion of conjectures and results in this direction.

Recently B.Dundas and R.McCarthy have proven that the stabilization of algebraic K-theory is naturally equivalent to topological Hochschild homology,

$$K^S(R; M) \simeq T(R; M)$$

for any simplicial ring R and any simplicial R-module M, *cf.* [4]. We note that both functors are defined for pairs $(L; P)$ where L is a functor with smash product and P is an L-bimodule; *cf.* [12]. An outline of a proof in this setting and by quite different methods, has been given by R.Schwänzl, R.Staffelt and F.Waldhausen. Hence the following result is a necessary condition for the conjecture mentioned above to hold.

Theorem. *Let L be a functor with smash product and P an L-bimodule. Then there is a natural weak equivalence,* $\mathrm{TC}^S(L; P)^{\wedge}_p \simeq T(L; P)^{\wedge}_p$.

It is not surprising that we have to p-complete in the case of TC since the cyclotomic trace is really an invariant of the p-completion of algebraic K-theory, *cf.* 1.4 below. The rest of this paragraph recalls cyclotomic spectra, topological Hochschild homology, topological cyclic homology and stabilization. In paragraph 2 we decompose topological Hochschild homology of a split extension of *FSP*'s and approximate TC in a stable range. Finally in paragraph 3 we study free cyclic objects and use them to prove the theorem.

Throughout G denotes the circle group, equivalence means weak homotopy equivalence and a G-equivalence is a G-map which induces an equivalence of H-fixed sets for any closed subgroup $H \leq G$.

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1.2. Let L be an *FSP* and let P be an L-bimodule. Then $\mathrm{THH}(L;P)_\bullet$ is the simplicial space with k-simplices

$$\underset{\overrightarrow{I^{k+1}}}{\mathrm{holim}}\, F(S^{i_0}\wedge\ldots\wedge S^{i_k}, P(S^{i_0})\wedge L(S^{i_1})\wedge\ldots\wedge L(S^{i_k}))$$

and Hochschild-type structure maps, *cf.* [12], and $\mathrm{THH}(L;P)$ is its realization. When $P=L$, considered as an L-bimodule in the obvious way, $\mathrm{THH}(L;L)$ is a cyclic space so $\mathrm{THH}(L;L)$ has a G-action. In both cases we use a thick realization to ensure that we get the right homotopy type, *cf.* the appendix. More generally if X is some space we let $\mathrm{THH}(L;P;X)_\bullet$ be the simplicial space

$$\underset{\overrightarrow{I^{k+1}}}{\mathrm{holim}}\, F(S^{i_0}\wedge\ldots\wedge S^{i_k}, P(S^{i_0})\wedge L(S^{i_1})\wedge\ldots\wedge L(S^{i_k})\wedge X),$$

where X acts as a dummy for the simplicial structure maps. If X has a G-action then $\mathrm{THH}(L;P;X)$ becomes a G-space and $\mathrm{THH}(L;L;X)$ a $G\times G$-space. We shall view the latter as a G-space via the diagonal map $\Delta\colon G\to G\times G$ and then denote it $\mathrm{THH}(L;X)$.

We define a G-prespectrum $t(L;P)$ in the sense of [9] whose 0'th space is $\mathrm{THH}(L;P)$. Let V be any orthogonal G-representation, or more precisely, any f.d. sub inner product space of a fixed 'complete G-universe' U. Then

$$t(L;P)(V)=\mathrm{THH}(L;P;S^V),$$

with the obvious G-maps

$$\sigma\colon S^{W-V}\wedge t(L;P)(V)\to t(L;P)(W)$$

as prespectrum structure maps. Here S^V is the one-point compactification of V and $W-V$ is the orthogonal complement of V in W. We also define a G-spectrum $T(L;P)$ associated with $t(L;P)$, *i.e.* a G-prespectrum where the adjoints $\tilde\sigma$ of the structure maps are homeomorphisms. We first replace $t(L;P)$ by a thickened version $t^\tau(L;P)$ where the structure maps σ are closed inclusions. It has as V'th space the homotopy colimit over suspensions of the structure maps

$$t^\tau(L;P)(V)=\underset{\overrightarrow{Z\subset V}}{\mathrm{holim}}\,\Sigma^{V-Z}t(L;P)(Z)$$

and as structure maps the compositions (t=t(L;P))

$$\Sigma^{W-V} \operatorname*{holim}_{\overrightarrow{Z\subset V}} \Sigma^{V-Z} t(Z) \cong \operatorname*{holim}_{\overrightarrow{Z\subset V}} \Sigma^{W-Z} t(Z) \to \operatorname*{holim}_{\overrightarrow{Z\subset W}} \Sigma^{W-Z} t(Z).$$

Here the last map is induced by the inclusion of a subcategory and as such is a closed cofibration, in particular it is a closed inclusion. Furthermore since V is terminal among $Z \subset V$ there is natural map $\pi: t^{\tau}(L;P) \to t(L;P)$ which is spacewise a G-homotopy equivalence. Next we define $T(L;P)$ by

$$T(L;P)(V) = \varinjlim_{W\subset U} \Omega^{W-V} t^{\tau}(L;P)(W)$$

with the obvious structure maps.

We can replace $\mathrm{THH}(L;P;S^V)$ by $\mathrm{THH}(L;S^V)$ above and get a G-prespectrum $t(L)$ and a G-spectrum $T(L)$. These possess some extra structure which allows the definition of $\mathrm{TC}(L)$ and we will now discuss this in some detail. For a complete account we refer to [6], see also [3].

1.3. Let C be a finite subgroup of G of order r and let J be the quotient. The r'th root $\rho_C: G \to J$ is an isomorphism of groups and allows us to view a J-space X as a G-space $\rho_C^* X$. Recall that the free loop space $\mathcal{L}X$ has the special property that $\rho_C \mathcal{L}X^C \cong_G \mathcal{L}X$ for any finite subgroup of G. Cyclotomic spectra, as defined in [3] and [6], is a class of G-spectra which have the analogous property in the world of spectra. This section recalls the defintion.

For a G-spectrum T there are two J-spectra T^C and $\Phi^C T$ each of which could be called the C-fixed spectrum of T. If $V \subset U^C$ is a C-trivial representation, then

$$T^C(V) = T(V)^C, \qquad \Phi^C T(V) = \varinjlim_{W\subset U} \Omega^{W^C - V} T(W)^C$$

and the structure maps are evident. There is a natural map $r_C: T^C \to \Phi^C T$ of J-spectra; $r_C(V)$ is the composition

$$T^C(V) \cong \varinjlim_{W\subset U} F(S^{W-V}, T(W))^C \xrightarrow{\iota^*} \varinjlim_{W\subset U} F(S^{W^C-V}, T(W)^C) = \Phi^C T(V)$$

where the map ι^* is induced by the inclusion of C-fixed points. The difference between T^C and $\Phi^C T$ is well illustated by the following example.

Example. Consider the case of a suspension G-spectrum $T = \Sigma_G^\infty X$,

$$T(V) = \varinjlim_{W\subset U} \Omega^{W-V}(S^W \wedge X).$$

We let $E_G H$ denote a universal H-free G-space, that is $E_G H^K \simeq *$ when $H \cap K = 1$ and $E_G H^K = \varnothing$ when $H \cap K \neq 1$. Then on the one hand we have the tom Dieck splitting

$$(\Sigma_G^\infty X)^C \simeq_J \bigvee_{H \leq C} \Sigma_J^\infty (E_{G/H}(C/H)_+ \wedge_{C/H} X^H),$$

and on the other hand the lemma shows that $\Phi^C(\Sigma_G^\infty X) \simeq_J \Sigma_J^\infty X^C$. Moreover the natural map $r_C\colon (\Sigma_G^\infty X)^C \to \Phi^C(\Sigma_G^\infty X)$ is the projection onto the summand $H = C$. $\square$

A J-spectrum D defines a G-spectrum $\rho_C^* D$. However this G-spectrum is indexed on the G-universe $\rho_C^* U^C$ rather than on U. To get a G-spectrum indexed on U we must choose an isometric isomorphism $f_C\colon U \to \rho_C^* U^C$, then $(\rho_C^* D)(f_C(V))$ is the V'th space of the required G-spectrum, which we denote it $\rho_C^\# D$.

We want the f_C's to be compatible for any pair of finite subgroups, that is the following diagram should commute

$$\begin{array}{ccc} U & \xrightarrow{f_{C_{rs}}} & \rho_{C_{rs}}^* U^{C_{rs}} \\ {\scriptstyle f_{C_r}}\downarrow & & \Vert \\ \rho_{C_r}^* U^{C_r} & \xrightarrow{\rho_{C_r}^*(f_{C_s})^{C_r}} & \rho_{C_r}^*(\rho_{C_s}^* U^{C_s})^{C_r}. \end{array}$$

Moreover the restriction of f_C to the G-trivial universe U^G induces an automorphism of U^G which we request be the identity. We fix our universe,

$$U = \bigoplus_{n\in\mathbb{Z},\alpha\in\mathbb{N}} \mathbb{C}(n)_\alpha,$$

where $\mathbb{C}(n) = \mathbb{C}$ but with G acting through the n'th power map. The index α is a dummy. Since $\rho_C^*\mathbb{C}(n) = \mathbb{C}(nr)$, where r is the order of C, we obtain the required maps f_C by identifying $\mathbb{Z} = r\mathbb{Z}$.

Definition. ([6]) A *cyclotomic spectrum* is a G-spectrum indexed on U together with a G-equivalence

$$\varphi_C\colon \rho_C^\# \Phi^C T \to T$$

for every finite $C \subset G$, such that for any pair of finite subgroups the diagram

$$\begin{array}{ccc} \rho_{C_r}^\# \Phi^{C_r} \rho_{C_s}^\# \Phi^{C_s} T & = & \rho_{C_{rs}}^\# \Phi^{C_{rs}} T \\ {\scriptstyle \rho_{C_r}^\# \Phi^{C_r} \varphi_{C_s}}\downarrow & & \downarrow{\scriptstyle \varphi_{C_{rs}}} \\ \rho_{C_r}^\# \Phi^{C_r} T & \xrightarrow{\varphi_{C_r}} & T \end{array}$$

commutes.

We prove in [6] that the topological Hochschild spectrum $T(L)$ defined above is a cyclotomic spectrum. The rest of this section recalls the definition of the φ-maps for $T(L)$. The definition goes back to [2] and begins with the concept of edgewise subdivision.