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# Algebraic $K$-Theory of operator ideals
## (after Mariusz Wodzicki)

by

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This is a report on some of Wodzicki's results on the algebraic $K$-theory of ideals in the ring of bounded operators on a Hilbert space. Unless indicated otherwise an ideal is always a two-sided ideal. We use the following notations. Let $H$ denote a countably infinite-dimensional Hilbert space, and let $\mathcal{B}(H)$ denote the algebra of bounded operators on $H$. It is known that the ordered set of proper ideals in $\mathcal{B}(H)$ has a maximal element, namely $\mathcal{K} = \mathcal{K}(H)$ of compact operators on $H$, and a minimal element, namely $\mathcal{F} = \mathcal{F}(H)$ of finite rank operators on $H$. These assertions are proved in Calkin [1941].

In Suslin-Wodzicki [1990] we find an isomorphism

$$K_*(B \otimes \mathcal{K}) \longrightarrow K_*^{\mathrm{top}}(B \otimes \mathcal{K}) = K_*^{\mathrm{top}}(B)$$

for every $C^*$-algebra $B$ where $\otimes$ denotes a $C^*$-completion of the algebraic tensor product. In LN 725, Karoubi conjectured that

$$K_*(B \widehat{\otimes}_\pi \mathcal{K}) \longrightarrow K_*^{\mathrm{top}}(B \widehat{\otimes}_\pi \mathcal{K})$$

is an isomorphism where $B$ is any Banach algebra with unit and $\widehat{\otimes}_\pi$ is Grothendieck's projective tensor product. The tensor product $B\widehat{\otimes}_\pi \mathcal{K}$ is not a $C^*$-algebra here even if $B$ is a $C^*$-algebra. The $C^*$-analogue of Karoubi's conjecture, that has been circulated among people working on $C^*$-algebras under the name of the Karoubi's conjecture, was proved in Suslin-Wodzicki [1990]. The original Karoubi conjecture has been proved by Wodzicki [unpublished]. When we put $B$ equal to the complex numbers $\mathbb{C}$, we have the isomorphism

$$K_m(\mathcal{K}) \longrightarrow K_m^{\mathrm{top}}(\mathbb{C}) = \begin{cases} \mathbb{Z} & \text{for even } m \\ 0 & \text{for odd } m. \end{cases}$$

The comparison map $K_m(\mathcal{K}) \longrightarrow K_m^{\text{top}}(\mathbb{C})$ factors through

$$K_m(\mathcal{B}(H), \mathcal{K}) \longrightarrow K_m^{\text{top}}(\mathbb{C})$$

where the relative $K$-groups $K_i(\mathcal{B}(H), \mathcal{K})$ for $i > 0$ are defined in the first section. Suslin and Wodzicki prove that the excision morphism $K_*(\mathcal{K}) \longrightarrow K_*(\mathcal{B}(H), \mathcal{K})$ is an isomorphism. In particular, they get the index isomorphism

$$\text{Ind} : K_{2i}(\mathcal{B}(H), \mathcal{K}) \longrightarrow \mathbb{Z} \quad \text{and} \quad K_{2i-1}(\mathcal{B}(H), \mathcal{K}) = 0$$

for all $i$. The isomorphism Ind is the classical index coming from the index of Fredholm operators.

In recent work M. Wodzicki studies other (two-sided) ideals $J \subset \mathcal{B}(H)$ and their algebraic $K$-theory $K_*(\mathcal{B}(H), J)$. He introduces a class of ideals $J$ in $\mathcal{B}(H)$ which, for the purpose of this paper, we call $B$-ideals, and he proves the following theorem which analyses index homomorphisms with values in cyclic homology for $B$-ideals.

(2.8) *Main exact index sequence.* There is an exact sequence functorial under inclusion $J \subset J'$ for $B$-ideals of the form

$$\begin{array}{ccccccc}
0 & \to & HC_{2j-1}(\mathcal{B}(H), J) & \to & K_{2j}(\mathcal{B}(H), J) & & \\
 & & & & \downarrow & & \\
 & & & & \mathbb{Z} & & \\
 & & & & \downarrow & & \\
 & & & & HC_{2j-2}(\mathcal{B}(H), J) & \to & K_{2j-1}(\mathcal{B}(H), J) \to 0.
\end{array}$$

Every Schatten ideal $\mathcal{C}_p$ is a $B$-ideal. For the definition of $B$-ideal, see (2.6).

In order to describe which ideals $J \subset \mathcal{B}(H)$ are $B$-ideals, we recall the definition of the power $J^r$ of $J$ to a strictly positive real number $r$ which generalizes the positive integer power $J^n$ of $J$. The real power $J^r$ of $J$ is the ideal generated by all $|A|^r$ for any $A \in J$ where $|A| = (AA^*)^{1/2}$. Observe that if $r \leq r'$, then we have the inclusion $J^r \supset J^{r'}$. We denote by $J_\infty = \bigcup_{r>0} J^r$, and one sees that $J_\infty$ is an ideal which contains $J$. We call the ideal $J_\infty$

the root completion of $J$. A $B$-ideal $J$ will be defined in terms of $J_\infty$ having certain properties. For this, see definition (2.6) and for the derivation of the five term sequence, see (2.7).

This text is part of a lecture given at the Strasbourg $K$-theory Conference, 1992. The author profitted from many discussions with Mariusz Wodzicki while we were both guests of the Forschergruppe, Topology and Noncommutative geometry, Heidelberg, May and June 1992 and during the preparation period of this text.

## §1. Definition of relative $K_*$ and $HC_*$ groups

In this section we review the definition of the relative $K_*$ and cyclic homology groups in order to fix notations and list the exact triangles that we will be using.

(1.1) *Definition.* Let $R$ be a $\mathbb{Z}$-algebra (with 1). The algebraic $K$-groups are $K_i(R) = \pi_i(BGL(R)^+)$, $i \geq 1$. Let $I$ be an ideal in $R$, let $R \to R/I$ denote the quotient morphism, and let $F(R,I)$ denote the homotopy fibre of the induced mapping on the plus constructions

$$BGL(R)^+ \longrightarrow B\overline{GL}(R/I)^+$$

where $\overline{GL}(R/I) = \mathrm{im}(GL(R) \to GL(R/I))$. The relative $K$-groups of the pair $(R,I)$ are the homotopy groups $K_i(R,I) = \pi_i(F(R,I))$, $i \geq 1$.

(1.2) *Definition.* We can extend the $K$-groups to negative degrees with the following recursive formulas

$$K_{-i}(R) = \mathrm{coker}\Big(K_{1-i}(R[t]) \oplus K_{1-i}(R[t^{-1}]) \to K_{1-i}(R[t,t^{-1}])\Big)$$

and

$$K_{-i}(I) = \ker(K_{-i}(\mathbb{Z} \ltimes I) \to K_{-i}(\mathbb{Z})) \text{ for } i > 0.$$

The relative $K_i(R,I)$ for $i \leq 0$ are defined either by the double ring construction as in Milnor's book or by the suspension functor as in Bass's book.

(1.3) *Remark.* From the fibre space exact triangle of homotopy groups we have the following exact triangle of $K$-groups

$$
\begin{tikzcd}
K_*(R,I) \arrow[rr] & & K_*(R) \arrow[dl] \\
& K_*(R/I) \arrow[ul, "-1"'] & \\
K_*(\mathbb{Z}\ltimes I, I) = K_*(I). \arrow[uu] & &
\end{tikzcd}
$$

The last terms of the exact sequence resulting from the fibre space homotopy sequence are the following

$$K_1(R) \rightarrow \overline{K}_1(R/I) = \pi_1(B\overline{GL}(R/I)^+) = \operatorname{im}(K_1(R) \rightarrow K_1(R/I)) \rightarrow 0.$$

The negative degree terms are exact from the nature of the definition of the relative $K$-groups in negative degrees.

The vertical morphism in this diagram leads to the following definition.

(1.4) *Definition.* An ideal $I$ satisfies $K_*$-excision provided the above vertical arrow $K_*(I) \rightarrow K_*(R,I)$ is an isomorphism for all rings $R$ containing $I$ up to isomorphism.

We know that $K_i(I) \rightarrow K_i(R,I)$ is always an isomorphism for $i \leq 0$.

(1.5) *Remark.* Let $I$ be an ideal satisfying $K_*$-excision. Then we have the following $\mathbb{Z}$-graded exact triangle

$$
\begin{tikzcd}
K_*(I) \cong K_*(R,I) \arrow[rr] & & K_*(R) \arrow[dl] \\
& K_*(R/I). \arrow[ul, "-1"'] &
\end{tikzcd}
$$

Now we consider the relative cyclic homology groups. Here we use the conventions of Wodzicki compatible with the usual conventions in $K$-theory,