MARC LEVINE

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Introduction

Bloch defined his higher Chow groups $\mathrm{CH}^q(-,p)$ in [B], with the object of defining an integral cohomology theory which rationally gives the weight-graded pieces $K_p(-)^{(q)}$ of K-theory. For a variety X, the higher Chow group $\mathrm{CH}^q(X,p)$ is defined as the pth homology of the complex $\mathcal{Z}^q(X,*)$, which in turn is built out of the codimension q cycles on $X \times \mathbb{A}^p$ for varying p, using the cosimplicial structure on the collection of varieties $\{X \times \mathbb{A}^p \mid p = 0, 1, \ldots\}$. In order to relate $\mathrm{CH}^q(X,p)$ with $K_p(X)$, Bloch used Gillet's construction of Chern classes with values in a Bloch-Ogus twisted duality theory [G]; this requires, among other things, that the complexes $\mathcal{Z}^q(X,*)$ satisfy a Mayer-Vietoris property for the Zariski topology, and that they satisfy a contraviant functoriality. Bloch attempted to prove the Mayer-Vietoris property by proving a localization theorem, identifying the cone of the restriction map

$$\mathcal{Z}(X,*) \to \mathcal{Z}(U,*),$$

for $U \to X$ a Zariski open subset of X, with the complex $\mathcal{Z}(X \backslash U,*)[1]$, up to quasi-isomorphism. There is a gap in Bloch's proof, which left open the localization property and the Mayer-Vietoris property for the complexes $\mathcal{Z}^q(X,*)$; essentially the same problem leaves a gap in the proof of contravariant functoriality. Recently, Bloch [B3] has provided a new argument which fills the gap in the proof of localization; this, together with a new argument for contravariant functoriality, should allow Bloch's original program for relating $\mathrm{CH}^q(X,p)$ with $K_p(X)$ to go through without further problem.

As part of the argument in [B], Bloch defined a map

$$\mathrm{CH}^q(X,p) \otimes \mathbb{Q} \to K_p(X)^{(q)} \tag{1}$$

for X smooth and quasi-projective over a field, relying on a λ-ring structure on relative K-theory with supports. It turns out that this approach can be followed and extended to show that the map (1) is an isomorphism, without relying on Chern classes (Theorem 3.1). An important new ingredient in this line of argument is the computation of certain relative K_0-groups in terms of the K_0 of an associated iterated double (see Theorem 1.10 and Corollary 1.11). A bit more work then enables us to prove the Mayer-Vietoris property (Theorem 3.3), a weak version of localization (essentially Poincaré duality) (Theorem 3.4), and contravariant functoriality (Corollary 4.9) for the rational complexes $\mathcal{Z}^q(X,*)\otimes\mathbb{Q}$. We also construct a product for the rational complexes $\mathcal{Z}^q(X,*)\otimes\mathbb{Q}$, and prove the projective bundle formula (Corollary 5.4). The arguments used in [B] then give rational Chern classes

$$c_{q,p}\colon K_{2q-p}(X)\to \mathrm{CH}^q(X,2q-p)\otimes\mathbb{Q},$$

satisfying the standard properties.

It turns out that it is somewhat more convenient to work with a modified version of $\mathcal{Z}^q(X,*)$, using a cubical structure rather than a simplicial structure. We show that the cubical complexes $\mathcal{Z}^q(X,*)^c$ are integrally quasi-isomorphic to the simplicial version $\mathcal{Z}^q(X,*)$ (Theorem 4.7), and have a natural exterior product in the derived category (see §5, especially Theorem 5.2). We also consider the "alternating" complexes $\mathcal{N}^q(k)$ defined by Bloch [B2], and used to construct a candidate for a motivic Lie algebra. We show that there is a natural quasi-isomorphism

$$\mathcal{Z}^q(\mathrm{Spec}(k),*)^c\otimes\mathbb{Q}\to\mathcal{N}^q(k)$$

(Theorem 4.11). The product structures are not quite compatible via this quasi-isomorphism; it is necessary to reverse the order of the product in one of the complexes to get a product-compatible quasi-isomorphism (Corollary 5.5).

The paper is organized as follows: We begin in §1 by proving some extensions of the results of Vorst on K_n-regularity, which we use to prove a basic result on the K_0-regularity of certain iterated doubles. We also recall some basic facts about relative K-theory, and use the K_0-regularity results to compute certain relative K_0 groups in terms of the usual K_0 of an iterated double. In §2 we use, following Bloch, the λ-operations on relative K-theory with supports to give a cycle-theoretic interpretation of certain relative K_0 groups, analogous to the classical Grothendieck-Riemann-Roch theorem relating the rational Chow ring to the rational K_0 for a smooth variety (see Theorem 2.7). In §3, we use this to show that Bloch's map

$$\mathrm{CH}^q(X,p)\otimes\mathbb{Q}\to K_p(X)^{(q)}$$

is an isomorphism for X smooth and quasi-projective. In §4, we relate the cubical complexes with Bloch's simplicial version, and also with his alternating version. In §5 we define products and prove the projective bundle formula for the rational complexes.

As a matter of notation, a *scheme* will always mean a separated, Noetherian scheme. For an abelian group A, we denote $A \otimes_{\mathbb{Z}} \mathbb{Q}$ by $A_{\mathbb{Q}}$; for a homological complex C_*, we denote the cycles in degree p by $Z_p(C_*)$, the boundaries by $B_p(C_*)$ and the homology by $H_p(C_*)$.

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§1. NK and relative K_0

In this section, we give a description of relative K_0, $K_0(X; Y_1, \ldots, Y_n)$, in terms of the K_0 of the so-called iterated double $D(X; Y_1, \ldots, Y_n)$, under certain assumptions on the scheme X and subschemes $Y_1, \ldots, Y_n$. We begin by extending some of Vorst's results on NK_p of rings (see [V]) to schemes over a ring.

Fix a commutative ring A, and let $\mathbf{Alg}_A$ denote the category of commutative A-algebras, $\mathbf{Ab}$ the category of abelian groups. For a ring R, let $p_R: R[T] \to R$ be the R-algebra homomorphism $p_R(T) = 0$. For a functor $F: \mathbf{Alg}_A \to \mathbf{Ab}$, let $NF: \mathbf{Alg}_A \to \mathbf{Ab}$ be the functor

$$NF(R) = \ker[F(p_R): F(R[T]) \to F(R)].$$

Define the associated functors N^qF for $q > 1$ inductively by

$$N^qF = N(N^{q-1}F).$$

We set $N^0F = F$.

For $R \in \mathbf{Alg}_A$ and $r \in R$, the R-algebra map

$$\begin{gathered}\phi_r: R[T] \to R[T] \\ \phi_r(T) = rT\end{gathered}$$

gives rise to the endomorphism $NF(\phi_r): NF(R) \to NF(R)$, thus $NF(R)$ becomes a $\mathbb{Z}[T]$-module with T acting via ϕ_r. Let $NF(R)_{[r]}$ denote the localization $\mathbb{Z}[T, T^{-1}] \otimes_{\mathbb{Z}[T]} NF(R)$. If r is a unit, then the map $NF(R) \to NF(R)_{[r]}$ is an isomorphism; letting R_r denote the localization of R with respect to the powers of r, the natural map

$$NF(R) \to NF(R_r)$$

factors canonically through $N(R)_{[r]}$:

$$\begin{tikzcd} NF(R) \arrow[rr] \arrow[dr] & & NF(R)_{[r]} \arrow[dl] \\ & NF(R_r) & \end{tikzcd}$$

For elements $r_1, \ldots, r_n$ of R, form the "augmented Čech complex"

(1.1)

$$0 \to NF(R) \xrightarrow{\epsilon} \bigoplus_{1 \le i \le n} NF(R_{r_i}) \to \ldots$$

$$\to \bigoplus_{1 \le i_0 < i_1 < \ldots < i_p \le n} NF(R_{r_{i_0}, r_{i_1}, \ldots, r_{i_p}}) \to \ldots \to NF(R_{r_1, \ldots, r_n}) \to 0.$$

where the map

$$\bigoplus_{1 \le i_0 < i_1 < \ldots < i_p \le n} NF(R_{r_{i_0}, r_{i_1}, \ldots, r_{i_p}}) \to \bigoplus_{1 \le i_0 < i_1 < \ldots < i_{p+1} \le n} NF(R_{r_{i_0}, r_{i_1}, \ldots, r_{i_{p+1}}})$$

is given as the direct sum over indices $(1 \le i_0 < i_1 < \ldots < i_{p+1} \le n)$ of the alternating sums:

$$\sum_{j=0}^{p+1} (-1)^j \delta_j : \oplus_{j=0}^{p+1} NF(R_{r_{i_0}, \ldots, \widehat{r_{i_j}}, \ldots, r_{i_{p+1}}}) \to NF(R_{r_{i_0}, \ldots, r_{i_{p+1}}}),$$