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HODGE-COMPONENTS OF CYCLIC HOMOLOGY FOR AFFINE QUASI-HOMOGENEOUS HYPERSURFACES

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0. Introduction

Let K be a field of characteristic zero, $R = K[X_1, \ldots, X_N]$ and the hypersurface $A = R/(F(X_1, \ldots, X_N))$ for a reduced polynomial $F \in R$. In [5] M. Gerstenhaber and S. Schack obtained a Hodge-decomposition of the Hochschild homology of commutative K-algebras A, where K is a field of characteristic zero. Using recent results by J. Majadas and A. Rodicio [12] and A. Lago and A. Rodicio [10], we are able to express the Hodge-components $HH_n^{(n-i)}(A, A)$ in terms of torsion submodules of exterior powers of $\Omega^1_{A/K}$, the module of Kaehler differentials. We find using the convention that $n + k = 2i$:

$$HH_n^{(i)}(A, A) \simeq \begin{cases} T(\Omega^k_{A/K}) & \text{for } i \neq n \ ; \\ \Omega^n_{A/K} & \text{for } i = n \ . \end{cases}$$

Moreover by Lemma 1, a straightforward generalization of theorem 4 in [14], we see $T(\Omega^k_{A/K}) = 0$ unless $codim\ Sing(A) < i < N + 1$. In [11] J.L.

Loday obtained a Hodge-decomposition of cyclic homology. We give an explicit formula for the Hodge-components of cyclic homology for reduced affine hypersurfaces defined by a quasi-homogeneous polynomial with an isolated singularity at the origin. Let $n + k = 2i$. Then in Theorems 2 and 3 we compute the Hodge-components of cyclic homology of a reduced hypersurface defined by a quasi-homogeneous polynomial. We get:

$$HC_n^{(i)} = \begin{cases} 0 & \text{for : } k \neq N-1 \ ; \\ T(\Omega^{N-1}_{A/K}) \simeq \Omega^N_{A/K} & \text{for } k = N-1 \ . \end{cases}$$

Thus I extend results of S. Geller, L. Reid and C. Weibel for curves [3] and of M.Vigué-Poirrier [15] for curves and surfaces defined by quasi-homogeneous polynomials with an isolated singularity at the origin. In [15] the author proves that for the reduced cyclic homology $\widetilde{HC}_n = \Omega^N_{A/K}$ or 0. In the last section we apply our results to $A - D - E$ singularities (c.f. [7] for a definition) and compute the dimension of the non-zero torsion submodules of some exterior power of the module of Kaehler differentials.

At this point the author would like to thank the organizers of the conference for providing me with an opportunity to present my work that is part of my Ph.D thesis at UC Berkeley under the supervision of Prof. M. Wodzicki. I also would like to thank Prof. C. Weibel and the referee for many helpful suggestions. After giving my talk I received a preprint by S. Geller and C. Weibel [4] that also computes the dimension of the Hodge-components of cyclic homology for hypersurfaces defined by a homogeneous polynomial.

There also is a paper by the Buenos-Aires group BACH in Advances of Mathematics [1] that contains some of my results using different methods.

1. Hodge-components of Hochschild homology

In [5] Corollary to Theorem 3.1 M. Gerstenhaber and S. Schack prove the following decomposition of $HH_n(A,A) := HH_n$ into A-modules for arbitrary commutative K-algebras A:

$$HH_n \simeq HH_n^{(1)} \oplus HH_n^{(2)} \oplus \ldots \oplus HH_n^{(n)}, \qquad \textit{with} \quad HH_n^{(n)} = \Omega^n_{A/K}.$$

Theorem 1: *Let $n+k=2i$. The Hodge-components of the Hochschild-homology modules of a reduced hypersurface A are given by:*

$$HH_n^{(i)}(A,A) \simeq \begin{cases} T(\Omega^k_{A/K}) & \textit{for } i \neq n \ ; \\ \Omega^n_{A/K} & \textit{for } i = n \ . \end{cases}$$

Proof: By M. Gerstenhaber and S.D Schack [5] p.231 we know that

$$HH_n^{(n-j)}(A,A) \simeq H_j(\bigwedge^{n-j} L_{A/K}),$$

where $L_{A/K}$ is the cotangent complex as defined in L. Illusie's book [8] II.1.2.3.1 p.123. By [8] Proposition III. 3.3.6. we have the following isomorphism in the derived category:

$$L_{A/K} \simeq (0 \to (F)/(F^2) \to \Omega^1_{R/K} \otimes_R A \to 0),$$

where $\Omega^1_{R/K} \otimes_R A$ sits in degree 0. For a definition of the terminology we refer to [8]. By [9] p.278 Corollaire 2.1.2.2 we have a canonical isomorphism in the derived category:

$$\bigwedge L_{A/K}[-*] = Kos^*(F/F^2 \xrightarrow{\phi} A \otimes \Omega^1_{R/K})$$

where $Kos^*(F/F^2 \xrightarrow{\phi} A \otimes \Omega^1_{R/K})$ denotes the Koszul complex of the A-module morphism $\phi : F/F^2 \to A \otimes \Omega^1_{R/K}$, given by: $\phi(F + (F^2)) = 1 \otimes_A dF$. By taking into account the shifting of degrees, we get:

$$H_j(\bigwedge^{n-j} L_{A/K}) \simeq H^{n-2j}(\bigwedge^{n-j} L_{A/K})[-n+j].$$

Moreover by a standard result of L. Illusie [9] p.278 corollaire 2.1.2.2., we have an isomorphism in the derived category

$$(\bigwedge^{n-j} L_{A/K})[-n+j] \sim Kos^*(\phi))_{n-j}.$$

So we get:

$$H^{n-2j}(\bigwedge^{n-j} L_{A/K})[-n+j] = H^{n-2j}(Kos^*(\phi)_{n-j}) =$$

$$H^{n-2j}((F^j)/(F^{j+1}) \otimes_A \Omega^{n-2j}_{R/K} \to (F^{j-1})/(F^j) \otimes_A \Omega^{n-2j+1}_{R/K}).$$

To compute

$$H^{n-2j}((F^j)/(F^{j+1}) \otimes_A \Omega^{n-2j}_{R/K} \to (F^{j-1})/(F^j) \otimes_A \Omega^{n-2j+1}_{R/K})$$