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COMPARISON THEOREM FOR λ-OPERATIONS IN HIGHER ALGEBRAIC K-THEORY

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Introduction

In his paper [G2], D. Grayson defined a map

$$\Lambda^k : \mathrm{Sub}_k\, G\mathcal{M} \to G^{(k)}\mathcal{M}$$

for every $k = 1, 2 \ldots$ and every exact category $\mathcal{M}$ with a suitable notion of exterior and tensor products, where both the domain and the codomain of the map are certain k-fold multisimplicial sets representing the homotopy type of K-theory of the category $\mathcal{M}$. This provides a definition of the operation λ^k on $K.\mathcal{M}$ as induced by the map Λ^k on the homotopy groups.

The G-construction $G\mathcal{M}$ is a simplicial set defined in [GG] whose vertices are in one-to-one correspondence with all pairs (A, B) of objects of $\mathcal{M}$, and an edge from (A, B) to (A', B') in $G\mathcal{M}$ is a pair of short exact sequences $(A \rightarrowtail A' \twoheadrightarrow C, B \rightarrowtail B' \twoheadrightarrow C)$ with equal cokernels. In order for the G-construction to serve as a domain of the map Λ^k, one should subdivide it first by the k-fold edgewise subdivision functor Sub_k : Simp.Sets $\to$ k-fold Multisimp.Sets (see [G2], sect. 4). The codomain $G^{(k)}\mathcal{M}$ is the iterated G-construction. Its vertices correspond bijectively to all 2^k-tuples of objects of $\mathcal{M}$ positioned naturally at the vertices of a k-dimensional cube.

Another definition of the operation λ^k in the same fashion was given by the author in [N]. This definition is provided by the map

$$\Lambda^k : \mathrm{Diag}\, \mathrm{Sub}_k\, G\mathcal{M} \to G(k; \mathcal{M})$$

where Diag : k-fold Multisimp.Sets $\to$ Simp.Sets is the total diagonal functor and $G(k; \mathcal{M})$ is a simplicial set whose vertices are in one-to-one correspondence with all $(k+1)$-tuples $(A_0, \ldots, A_k)$ of objects of $\mathcal{M}$, and an edge from (A_i) to (B_i) is a $(k+1)$-tuple of short exact sequences $A_i \rightarrowtail B_i \twoheadrightarrow C_i)$ together with a long exact sequence of cokernels $0 \to C_k \to C_{k-1} \to \cdots \to C_0 \to 0$.

It is not hard (and this is done in [G2] and [N]) to establish independently the equivalence of each of the two definitions of λ-operations to that given by means of Quillen's plus construction in [Hi] and [Kr] in the case of K-theory of a ring. The purpose of this paper is to show directly the equivalence of the above two definitions for any exact category $\mathcal{M}$ with operations, in the very spirit of the definitions, i.e., by means of certain simplicial structures.

We demonstrate the idea in the case $k = 2$. The map Λ^2 of Grayson takes a vertex (V, W) of $G\mathcal{M}$ to the 4-tuple of objects of $\mathcal{M}$

$$\begin{pmatrix} W \wedge W & W \otimes W \\ V \wedge V & V \otimes W \end{pmatrix}$$

regarded as a vertex of $G^{(2)}\mathcal{M}$. We observe that the objects $W \wedge W$ and $W \otimes W$ standing in the 1st row admit the natural maps $W \wedge W \rightarrowtail W \otimes W$ and $W \otimes W \twoheadrightarrow W \wedge W$, which lead to the pair of dual short exact sequences $0 \to W \wedge W \to W \otimes W \to S^2W \to 0$ and $0 \leftarrow W \wedge W \leftarrow W \otimes W \leftarrow W \circ W \leftarrow 0$, where S^2W is the symmetric square and $W \circ W = D^2W$ is the 2nd divided power (invariants of the symmetric group Σ_2 action on $W \otimes W$ in the case of modules). In the sequel, we prefer to use sequences of the second type, i.e. coSchur complexes (*cf.* [ABW], ch. V).

This observation leads to the definition of a bisimplicial set $A(2; \mathcal{M})$ whose vertices are in one-to-one correspondence with all diagrams of the type

$$\begin{pmatrix} A & \twoheadleftarrow & B & \leftarrowtail & C \\ D & & E & & \end{pmatrix}$$

where the sequence $A \twoheadleftarrow B \leftarrowtail C$ is exact. There is a natural map $A(2; \mathcal{M}) \to G^{(2)}\mathcal{M}$ given on vertices by

$$\begin{pmatrix} A & \twoheadleftarrow & B & \leftarrowtail & C \\ D & & E & & \end{pmatrix} \longmapsto \begin{pmatrix} A & B \\ D & E. \end{pmatrix}$$

This map is a homotopy equivalence, and the map $\Lambda^2 : Sub_2 G\mathcal{M} \to G^{(2)}\mathcal{M}$ from [G2] can be lifted to a map $\Lambda^2 : \mathrm{Sub}_2\, G\mathcal{M} \to A(2; \mathcal{M})$,

$$(V, W) \longmapsto \begin{pmatrix} W \wedge W & \leftarrow & W \otimes W & \leftarrow & W \circ W \\ V \wedge V & & V \otimes W & & \end{pmatrix}.$$

On the other hand, there is a natural map $\mathrm{Diag}\, A(2; \mathcal{M}) \to G(2; \mathcal{M})$ given on vertices by

$$\begin{pmatrix} A & \twoheadleftarrow & B & \leftarrowtail & C \\ D & & E & & \end{pmatrix} \longmapsto (D, E, C)\ ,$$

which also proves to be a homotopy equivalence, and the composite map $\operatorname{Diag}\operatorname{Sub}_2 G\mathcal{M} \to \operatorname{Diag} A(2;\mathcal{M}) \to G(2;\mathcal{M})$ given on vertices of $G\mathcal{M}$ by

$$(V,W) \mapsto \begin{pmatrix} W\wedge W & \twoheadleftarrow & W\otimes W & \leftarrowtail & W\circ W \\ V\wedge V & & V\otimes W & & \end{pmatrix} \mapsto (V\wedge V,\ V\otimes W,\ W\circ W)$$

is nothing but the map Λ^2 from [N]. Thus, we result with the equivalence of the Λ^2-maps of [G2] and [N].

For an arbitrary k, we define a k-fold multisimplicial set $A(k;\mathcal{M})$ and a map $\Lambda^k : \operatorname{Sub}_k G\mathcal{M} \to A(k;\mathcal{M})$ by means of which the Λ-maps of [G2] and [N] can be linked in the following manner.

MAIN THEOREM. — *The commutative diagram of spaces holds,*

$$\begin{array}{ccccc}
|\operatorname{Sub}_k G\mathcal{M}| & \xrightarrow{\Lambda^k_{[G2]}} & |G^{(k)}\mathcal{M}| & & \\
\| & & \uparrow & & \\
|\operatorname{Sub}_k G\mathcal{M}| & \xrightarrow{\Lambda^k \text{loc.cit.}} & |A(k;\mathcal{M})| & \cong & |\operatorname{Diag} A(k;\mathcal{M})| \\
\|\, \wr & & & & \downarrow \\
|\operatorname{Diag}\operatorname{Sub}_k G\mathcal{M}| & & \xrightarrow{\qquad\Lambda^k_{[N]}\qquad} & & |G(k;\mathcal{M})|
\end{array}$$

where all arrows are given by certain simplicial maps, the vertical arrows on the right are homotopy equivalences, the map Λ^k on the top is that of [G2], and Λ^k at the bottom is defined in [N].

REMARK. The construction of each of the three arrows in the bottom square depends on choice of cokernels for all admissible monomorphisms in $\mathcal{M}$, hence these maps are defined up to natural simplicial homotopy. Given such a choice, the bottom square is strictly commutative.

In section 1 we recall the definition of multidimensional $S.$ and C (mapping cone) constructions given in [G3] , and develop some technique for them. This technique is based mainly on a generalization for multidimensional case of the Theorem C of Grayson [G1]. It enables one to compute the C-construction of a cube of exact categories as a homotopy fibre of the map of $S.$ constructions of the corresponding faces, under a certain assumption on the cube (*cf.* Proposition 1.6). In order to prove such a generalization, we need to restrict the class of dominant functors introduced in [G1] and consider strictly dominant functors (*cf.* Definition 1.1) which prove to be stable under applying the C-construction degreewise (Proposition 1.5).

In section 2, some finite categories of words (actually ordered sets) are introduced. We declare certain sequences of words to be "long exact" and call them "formal Schur complexes". Then we define some exact categories of diagrams in $\mathcal{M}$ in which the positions correspond to those words, and formal Schur complexes give rise to long exact sequences in those diagrams. We define the multisimplicial set $A(k;\mathcal{M})$ by means of these categories of diagrams via the C-construction and show that the natural forgetful map $A(k;\mathcal{M}) \to G^{(k)}\mathcal{M}$ is a homotopy equivalence. Hence, $A(k;\mathcal{M})$ represents the $K.\mathcal{M}$ homotopy type.

In section 3, we construct a chain of k-fold multisimplicial sets $A(k;\mathcal{M}) = A(k,k;\mathcal{M}) \to A(k;k-1;\mathcal{M}) \to \cdots \to A(k,1;\mathcal{M})$ and show that all maps in it are homotopy equivalences. Then we define a map from the total diagonal $\operatorname{Diag} A(k,1;\mathcal{M})$ to $G(k;\mathcal{M})$ which also proves to be a homotopy equivalence. This results with a homotopy equivalence of simplicial sets $\operatorname{Diag} A(k;\mathcal{M}) \to G(k;\mathcal{M})$.

In section 4 we define a map $\Lambda^k : \operatorname{Sub}_k G\mathcal{M} \to A(k;\mathcal{M})$ with "real" coSchur complexes corresponding to formal Schur complexes in the diagrams. We check that this map is compatible with the Λ-maps defined in [G2] and [N], therefore establishing the desired equivalence.

We note that another (with respect to $A(k,\ell;\mathcal{M})$) interesting class of multisimplicial spaces representing the delooping of the K-theory homotopy type was introduced by Grayson in [G3] in order to define the operations of Adams. Further investigation of operations in higher K-theory on the simplicial level is carried out in the preprint of B. Köck [Kö].

§1. Simplicial Technique

We recall the definition of the multidimensional C and $S.$ constructions (*cf.* [G3], sect. 4).

Let Δ denote the category of finite nonempty totally ordered sets and nondecreasing maps. For any partially ordered set P, we denote by $\operatorname{Ar} P$ the category of arows in P, where a map of arrows is an obvious commutative diagram. We use the notation i/j for the arrow $(j \leq i) \in \operatorname{Ar} P$. Given an exact category $\mathcal{M}$ with a distinguished zero object $*$, the S-construction of Waldhausen is a simplicial set $S.\mathcal{M}$ with $S.\mathcal{M}[P] = \operatorname{Exact}(\operatorname{Ar P}, \mathcal{M})$, $P \in \Delta$, with obvious face and degeneracy maps, where "Exact" refers to