CLAUDIO PEDRINI
CHARLES WEIBEL

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DIVISIBILITY IN THE CHOW GROUP OF ZERO-CYCLES ON A SINGULAR SURFACE

by

Claudio PEDRINI[1] and Charles WEIBEL[2]

§0. Introduction.

In this paper we study the divisibility of the Chow group $CH^2(X)$ of 0-cycles on a surface X over a field k. When X is smooth this question has been studied by several authors [MSw] [B2] [R] [CT-R], and we extend many of their results to singular surfaces.

The Chow group of a singular surface X is defined as follows. Choose a closed $Y \subset X$ containing the singular locus of X but no irreducible component of X, and let $Z^2(X,Y)$ be the free abelian group on the set of codimension 2 points of $X - Y$. For each closed curve T in X missing Y, and every rational function f on T, the divisor (f) should equal 0 in $CH^2(X)$. If $\dim Y = 0$, $CH^2(X) = CH^2(X,Y)$ is the quotient of $Z^2(X,Y)$ by the subgroup spanned by these divisors; it is independent of Y because by [PW1, 2.2] it is isomorphic to $SK_0(X)$, the subgroup of $K_0(X)$ consisting of elements of rank 0 and determinant 1. If $\dim Y = 1$ we form $CH^2(X) = CH^2(X,Y)$ by adding the extra relations that $(f) = 0$ for every closed curve T on X which is locally cut out by a nonzerodivisor and every $f \in k(T)$ such that the support of (f) misses $T \cap Y$; this group is also independent of Y, because by [LW] we have $CH^2(X,Y) \cong SK_0(X)$.

If X is a surface and $\mathcal{K}_2$ denotes the Zariski sheaf associated to the presheaf $U \mapsto K_2(U)$, there is a well known isomorphism, called "Bloch's Formula" :

$$(0.1) \qquad CH^2(X) \cong SK_0(X) \cong H^2_{\mathrm{zar}}(X, \mathcal{K}_2).$$

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It was discovered by Bloch [B1] for smooth quasiprojective surfaces, extended to all smooth varieties by Quillen [Q], and to singular surfaces by Levine [L1]; see also [PW1, 8.9]. For regular surfaces, (0.1) also follows from the Brown-Gersten spectral sequence [BG]. For general 2-dimensional noetherian schemes, (0.1) follows from Thomason's generalization [TT, 10.3] of the Brown-Gersten spectral sequence.

Our results relate $CH^2(X)$ to the Zariski cohomology of a certain sheaf $\mathcal{H}^2$ on X. To define it, fix an integer n such that $\frac{1}{n} \in k$, let μ_n denote the étale sheaf of n^{th} roots of unity, and set $\mu_n^{\otimes 2} = \mu_n \otimes \mu_n$. By definition, $\mathcal{H}^2 = \mathcal{H}^2(\mu_n^{\otimes 2})$ is the Zariski sheaf associated to the presheaf $U \mapsto H^2_{\text{et}}(U, \mu_n^{\otimes 2})$ of étale cohomology. Since this sheaf has exponent n, it is convenient to adopt the notation that G/n denotes G/nG and ${}_nG$ denotes $\{x \in G : nx = 0\}$ for any abelian group or sheaf G. Here is our first result.

THEOREM A. — *Let X be a quasiprojective surface over a field k containing $\frac{1}{n}$. Then the Chern class $c_{2,2} : K_2(U) \to H^2_{\text{et}}(U, \mu_n^{\otimes 2})$ induces an isomorphism :*

$$CH^2(X)/n \cong H^2_{\text{zar}}(X, \mathcal{K}_2)/n \cong H^2_{\text{zar}}(X, \mathcal{H}^2(\mu_n^{\otimes 2}))$$

This result was originally proven in the smooth case by Bloch and Ogus [BO], and generalized to the case of isolated singularities by Barbieri-Viale [BV1, 3.9]. We give a short proof of Theorem A in §1, using the Nisnevich topology on X, a method suggested to us by R. Thomason.

After submitting this paper, which contained a second more technical proof of Theorem A in §2, we became aware of the following unpublished result of Ray Hoobler [Hoob] which, given Bloch's formula (0.1), immediately implies Theorem A.

HOOBLER'S THEOREM 0.2. — *Let k be a field containing $\frac{1}{n}$.*

1) If A is a semilocal ring, essentially of finite type over k, then the Chern class $c_{22} : K_2(A) \to H^2_{\text{et}}(A, \mu_n^{\otimes 2})$ is an isomorphism.

2) If X is a quasiprojective scheme over k, there is an isomorphism of (Zariski) sheaves

$$c_{2,2} : \mathcal{K}_2/n \to \mathcal{H}^2(\mu_n^{\otimes 2}).$$

When X or A is smooth over k, this theorem is implicit in Merkurjev and Suslin's work [MS, §18]; see [B3, 3.3] [CT-R, p.168] and [PW2, 4.3]. When X is a singular curve, this theorem was proven in [PW2, 5.2].

Our original proof of Theorem A is therefore obsolete. As a favor to the reader, we have deleted it. It was the original §2 of this paper.

The current §2 gives a short survey of the étale Chern classes c_{ij}. We also prove that the isomorphism in Theorem A lifts Grothendieck's Chern class $c_{2,4} : K_0(X) \to H^4_{\text{et}}(X, \mu_n^{\otimes 2})$ to $SK_0(X)$ in the sense that $c_{2,4}$ is the composite

$$SK_0(X) \twoheadrightarrow SK_0(X)/n \cong H^2_{\text{zar}}(X, \mathcal{H}^2) \xrightarrow{\gamma} H^4_{\text{et}}(X, \mu_n^{\otimes 2}),$$

γ being the edge map in the Leray spectral sequence for $X_{\text{et}} \to X_{\text{zar}}$. When X is smooth this proves that the "cycle map" considered in [CT-R] and [Sai,§5] is just $c_{2,4}$.

In §3 we consider the normalization $\pi : \tilde{X} \longrightarrow X$ of X. Using Mayer-Vietoris sequences, we relate $CH^2(X)/n$ to the Chow group $CH^2(\tilde{X})/n$. Let Y denote the singular locus of X, and set $\tilde{Y} = \pi^{-1}(Y)$, so that we have a cartesian square :

$$\begin{tikzcd} \widetilde{Y} \arrow[r, "j"] \arrow[d] & \widetilde{X} \arrow[d, "\pi"] \\ Y \arrow[r, "i"] & X. \end{tikzcd}$$

THEOREM B. — *Assume that k contains μ_n and $\frac{1}{n}$. Then there is an exact sequence for the sheaf $\mathcal{H}^2 = \mathcal{H}^2(\mu_n^{\otimes 2})$:*

$$H^1(\widetilde{X}, \mathcal{H}^2) \oplus H^1(Y, \mathcal{H}^2) \to H^1(\widetilde{Y}, \mathcal{H}^2) \to H^2(X, \mathcal{H}^2) \to H^2(\widetilde{X}, \mathcal{H}^2) \to 0.$$

Using Theorem A and the two isomorphisms $H^1(Y, \mathcal{H}^2) \cong SK_1(Y)/n$ and $H^1(\tilde{Y}, \mathcal{H}^2) \cong SK_1(\tilde{Y})/n$ of [PW2, 5.1], we can restate Theorem B as follows.

COROLLARY C. — *With n as in Theorem B, there is an exact sequence :*

$$H^1(\tilde{X}, \mathcal{H}^2) \oplus SK_1(Y)/n \to SK_1(\tilde{Y})/n \to CH^2(X)/n \to CH^2(\tilde{X})/n \to 0.$$

In the Appendix, we indicate how much of Corollary C can be obtained from pure K-theoretic techniques, i.e., without resorting to $\mathcal{H}^2$.

In §4 we relate the n-torsion in the Chow group of $\tilde{X}$ to the term $H^1(\tilde{X}, \mathcal{H}^2)$ appearing in Corollary C, as well as to the quotient $H^1(\tilde{X}, \mathcal{K}_2)$ of $SK_1(\tilde{X})$. When X is smooth, we know by [B3, 1.12][MS, 8.7.8(e)] that there is an exact sequence :

$$(0.3) \qquad 0 \to H^1(X, \mathcal{K}_2)/n \to H^1(X, \mathcal{H}^2(\mu_n^{\otimes 2})) \to {}_nCH^2(X) \to 0.$$

When X is a surface with isolated singularities, (0.3) needs to be modified because the subsheaf ${}_n\mathcal{K}_2$ of n-torsion elements in $\mathcal{K}_2$ has more complicated cohomology. Indeed, the vanishing of $H^2(X, {}_n\mathcal{K}_2)$ in the smooth case is the basis for the proof of (0.3) in [MS], but if X has isolated singularities we show in 4.2 that

$$H^2(X, {}_n\mathcal{K}_2) \cong H^2(X, \mathcal{H}^1(\mu_n^{\otimes 2})).$$

This group is just $H^2(X, \mathcal{O}_X^*)/n$ when $\mu_n \subset k$, and we know that it can be nonzero for normal surfaces; see [PW1, 5.9]. We are able to prove the following generalization of (0.3) in §4. (Again, we have deleted those parts which Hoobler's Theorem makes obsolete.)

THEOREM D. — *Let X be a quasiprojective surface over a field k containing $\frac{1}{n}$. Assume that X is normal, or more generally that* $\mathrm{Sing}(X)$ *is finite. Then there is an exact sequence :*

$$H^0(X, \mathcal{K}_2/n) \xrightarrow{\gamma} H^2(X, {}_n\mathcal{K}_2) \to H^1(X, \mathcal{K}_2)/n \to H^1(X, \mathcal{K}_2/n) \to {}_nCH^2(X) \to 0$$

Remark. Presumably the map $H^0(X, \mathcal{H}^2) \xrightarrow{\gamma} H^2(X, \mathcal{H}^1)$ in Theorem D is the differential in the Leray spectral sequence converging to $H^*_{\mathrm{et}}(X, \mu_n^{\otimes 2})$. If so, and we write $NH^3(X)$ for the kernel of $H^3_{\mathrm{et}}(X, \mu_n^{\otimes 2}) \to H^0(X, \mathcal{H}^3)$, then we may restate Theorem D as the following exact sequence, which generalizes part of the sequence of [Suslin, 4.4].

$$(0.4) \qquad 0 \to H^1(X, \mathcal{K}_2)/n \to NH^3(X) \to {}_nCH^2(X) \to 0$$

COROLLARY E (Collino [C]). — *Suppose that k is either an algebraically closed field, or the reals $\mathbb{R}$, or a local field. Let X be a surface having only isolated singularities. Then the n-torsion in $CH^2(X)$ is finite for every n with $\frac{1}{n} \in k$.*

Proof. Fix n and let k be any field such that $H^i_{et}(k, M)$ is finite for constructible n-torsion sheaves M. Then each $H^q_{et}(X, \mu_n^{\otimes i})$ is finite by [SGA4, XVI.5.1]. When X is a surface, the Leray spectral sequence $H^p(X, \mathcal{H}^q) \Rightarrow H^*_{et}(X, \mu_n^{\otimes i})$ degenerates enough to show that the group $H^1(X, \mathcal{H}^2(\mu_n^{\otimes 2})) = H^1(X, \mathcal{K}_2/n)$ is finite. Now apply Theorem D. $\square$

There is a "degree" map $CH^2(X) \to \mathbb{Z}^c$, where c denotes the number of irreducible proper components of X. The image Λ has finite index in $\mathbb{Z}^c$, and $CH^2(X) \cong \Lambda \oplus A_0(X)$, where $A_0(X)$ is the group of zero cycles of relative "degree" zero. Therefore all of our divisibility results are actually statements about the divisibility of the subgroup $A_0(X)$ of $CH^2(X)$.