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1 Introduction.

In [MS, Theorem 1], M. P. Murthy–R. G. Swan have shown that a stably free projective module over a two-dimensional affine variety A over an algebraically closed field k is free. (The example of the tangent bundle over the real two sphere shows that the condition that the base field is algebraically closed is necessary.) They showed that every unimodular vector $(a, b, c) \in Um_3(A)$ could be transformed to a unimodular vector of the form (x^2, y, z), and that a unimodular vector of the form (x^2, y, z) over any commutative ring A can always be completed to an invertible matrix. In [Su1] A. Suslin generalised this by showing that a unimodular vector of the form $(a_0, a_1, a_2^2, \ldots, a_r^r)$ over any commutative ring A can always be completed to an invertible matrix; from which he deduced that a stably free projective A-module of rank $\geq \dim(A)$ is free where A is an affine algebra over an algebraically closed field k. In [Su2] Suslin generalised this further by proving that stably free projective A-modules of rank $\geq \dim(A)$ are free when A is an affine algebra over a perfect C_1 field k. (Whenever we speak of "perfect C_1 field", which is admittedly not a very useful combination outside characteristic 0, the more technical conditions in 3.1 actually suffice.) In this note we prove a K_1 analogue of Suslin's result. We prove that:

Theorem 1 (cf. 3.4) *Let A be a non-singular affine algebra of dimension $d \geq 2$ over a perfect C_1-field k. Then*

$$SL_{d+1}(A) \cap E_{d+2}(A) = E_{d+1}(A)$$

i.e. a stably elementary $\sigma \in SL_{d+1}(A)$ belongs to $E_{d+1}(A)$. Consequently, the natural map

$$SL_r(A)/E_r(A) \longrightarrow SK_1(A)$$

is an isomorphism for $r \geq d + 1$.

A beautiful theorem of L. N. Vaserstein [SV, Corollary 7.4], identifies $Um_3(A)/E_3(A)$, the coset space of unimodular 3-vectors $Um_3(A)$ modulo action of the Elementary matrices $E_3(A)$, with the Symplectic Elementary Witt group $W_E(A)$ when $\dim(A) \leq 2$. The correspondence

$$V : Um_3(A)/E_3(A) \to W_E(A)$$

which he has defined for any commutative ring A, is known as the *Vaserstein symbol.* As a consequence of the above theorem we obtain that the Vaserstein symbol is also an isomorphism if A is a three dimensional non-singular affine algebra over a perfect C_1 field k.

We present an example, based on the Hopf map $S^3 \to S^2$, to show that some condition on the field is again necessary.

For higher dimensional rings it was shown by the second author that one still has an (abelian) group structure on the orbit set $Um_d(A)/E_d(A)$, even though one no longer possesses such a nice interpretation as in Vaserstein's theorem. Following Suslin [Su4] one would now like to have a homomorphism from $SL_d(A)$ to this group $WMS_d(A) \approx Um_d(A)/E_d(A)$. It was shown in [vdK2, Prop. 7.10] that this will not work in general, but as another byproduct of the above we show that things improve over perfect C_1 fields.

Theorem 2 *Let A be a non-singular affine algebra of dimension $d \geq 3$ over a perfect C_1-field k. Then the "first row map"*

$$SL_d(A) \to WMS_d(A)$$

is a homomorphism. Taking its cokernel provides a group structure on $Um_d(A)/SL_d(A)$.

If in our C_1 field -1 is a square, one sees from [Ra2, (1.3)] that the row $(a_0, a_1, a_2^2, \ldots, a_{d-1}^{d-1})$—completable by Suslin—represents the $(d-1)!$ power of the class of $(a_0, a_1, a_2, \ldots, a_{d-1})$. So the group $Um_d(A)/SL_d(A)$ in the theorem is a torsion group of exponent at most $(d-1)!$ This makes us hope that it will be manageable in some cases.

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2 Preliminaries on the Vaserstein symbol.

All rings considered are commutative with 1. The set $M_{r,s}(A)$ consists of all matrices of size $r \times s$ over A. Write $M_r(A) = M_{r,r}(A)$.

A vector $\boldsymbol{v} = (v_1, v_2, \ldots, v_r) \in A^r$ is said to be *unimodular* if there are elements $w_1, \ldots, w_r$ in A such that $v_1w_1 + \ldots + v_rw_r = 1$. The set of all unimodular vectors $\boldsymbol{v} \in A^r$ will be denoted $Um_r(A)$. The standard basis of A^r is written $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_r$.

The group $SL_r(A)$ of invertible matrices of determinant 1 acts on A^r in a natural way:

$$\sigma : \boldsymbol{v} \mapsto \boldsymbol{v}\sigma,$$

if $\boldsymbol{v} \in A^r$, $\sigma \in SL_r(A)$. This map preserves $Um_r(A)$, so $SL_r(A)$ acts on $Um_r(A)$.

Let $E_r(A)$ denote the subgroup of $SL_r(A)$ consisting of all *elementary* matrices, i.e. those matrices which are a finite product of the elementary generators $E_{ij}(\lambda) = I_r + e_{ij}(\lambda)$, $1 \le i \ne j \le r$, $\lambda \in A$, where $e_{ij}(\lambda) \in M_r(A)$ has entry λ in its (i,j)-th position, and all other entries zero. Thus $E_r(A)$ acts on $Um_r(A)$; if $\boldsymbol{v}, \boldsymbol{w} \in Um_r(A)$, let $\boldsymbol{v} \sim_E \boldsymbol{w}$ mean that $\boldsymbol{v} = \boldsymbol{w}\epsilon$ for some $\epsilon \in E_r(A)$. Let $Um_r(A)/E_r(A)$ be the set of equivalence classes of vectors $\boldsymbol{v} \in Um_r(A)$ under the equivalence $\sim_E$; and let $[\boldsymbol{v}] = [v_1, \ldots, v_r]$ denote the equivalence class of $\boldsymbol{v} = (v_1, \ldots, v_r)$.

If $\alpha \in M_r(A)$, $\beta \in M_s(A)$, then $\alpha \perp \beta$ denotes $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \in M_{r+s}(A)$. An alternating matrix ϕ has diagonal entries 0 and is skew-symmetric, i.e. transpose$(\phi) = -\phi$. We can define inductively an alternating matrix $\psi_r \in E_{2r}(\mathbb{Z})$, by setting, for $r \ge 2$,

$$\psi_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \psi_r = \psi_{r-1} \perp \psi_1.$$

For an alternating matrix $\phi \in M_{2r}(A)$ its determinant $\det(\phi)$ is a square $(pf(\phi))^2$ of a polynomial $pf(\phi)$ (called the Pfaffian) in the matrix entries with coefficients ± 1. An odd sized alternating matrix has Pfaffian 0, and, clearly, on even sized alternating matrices it is defined up to a sign, to fix which we insist that $pf(\psi_r) = 1$, for all r. For instance, if $\boldsymbol{v} = (v_0, v_1, v_2)$, $\boldsymbol{w} = (w_0, w_1, w_2)$, then

$$V(\boldsymbol{v}, \boldsymbol{w}) = \begin{pmatrix} 0 & v_0 & v_1 & v_2 \\ -v_0 & 0 & w_2 & -w_1 \\ -v_1 & -w_2 & 0 & w_0 \\ -v_2 & w_1 & -w_0 & 0 \end{pmatrix}$$

has Pfaffian $v_0w_0 + v_1w_1 + v_2w_2$. In particular, if $v_0w_0 + v_1w_1 + v_2w_2 = 1$, then $V(\boldsymbol{v}, \boldsymbol{w})$ has Pfaffian 1.

If $\alpha \in M_{2r}(A)$, and ϕ, θ are alternating matrices, then it can be checked that $pf(\alpha^t\phi\alpha) = pf(\phi)\det(\alpha)$, and that $pf(\phi \perp \theta) = pf(\phi)\,pf(\theta)$.

Two alternating matrices $\alpha \in M_{2r}(A)$, $\beta \in M_{2s}(A)$ are said to be *equivalent w.r.t.* $E(A)$ if there is a $\epsilon \in E_{2(r+s+l)}(A)$, for some l, such that

$$\alpha \perp \psi_{s+l} = \epsilon(\beta \perp \psi_{s+l})\epsilon^t.$$

It can be seen, cf. [SV, p. 945], that $\perp$ induces the structure of an abelian group on the set of all equivalence classes of alternating matrices with Pfaffian 1. This group is called the Symplectic Elementary Witt group and is denoted by $W_E(A)$.

The *Vaserstein symbol* $V = V_A : Um_3(A)/E_3(A) \longrightarrow W_E(A)$ is the map

$$[(a,b,c)] \mapsto [V(\boldsymbol{v},\boldsymbol{w})],$$

where $\boldsymbol{v} = (a,b,c)$, $\boldsymbol{w} = (a',b',c')$ with $aa' + bb' + cc' = 1$. In [SV, Theorem 5.2], L. N. Vaserstein has shown that this map is well defined (i.e. it is independent of both the choice of representative $\boldsymbol{v}$ in $[(a,b,c)]$, as well as the choice of a',b',c' such that $aa' + bb' + cc' = 1$).

Recall that sdim(A) stands for the stable range dimension of A, i.e. one less than the stable rank $sr(A)$ of [Va2]. For noetherian A it does not exceed the Krull dimension $\dim(A)$.

Lemma 2.1 *Let A be a commutative ring for which $Um_r(A) = e_1E_r(A)$, for $r \geq 5$. Then V_A is surjective. If, moreover, $SL_4(A) \cap E(A) = E_4(A)$ then V_A is bijective. In particular, if* sdim$(A) \leq 3$, *and $SL_4(A) \cap E_5(A) = E_4(A)$, then V_A is bijective.*

Proof: L. N. Vaserstein in [SV, Theorem 5.2(c)] has shown that V_A is surjective. Apply [SV, Lemma 5.1] to conclude that V_A is injective if $SL_4(A) \cap E(A) = E_4(A)$. L. N. Vaserstein's stability estimate for the linear group in [Va1] settles the last assertion. □

3 Decrease in the injective stability estimate for K_1 of a regular affine algebra over a C_1 field

In [Va1] L. N. Vaserstein shows that

$$\frac{SL_r(A)}{E_r(A)} = \frac{SL_{r+1}(A)}{E_{r+1}(A)} = \cdots = SK_1(A),$$

where $SK_1(A)$ is the Whitehead group of A, when $r \geq \max\{3, d+2\}$, $d =$ sdim(A).

The reader may construct or find examples (cf. [vdK2, Prop. 7.10]) of regular affine algebras A over the *real* numbers $\mathbb{R}$ for which $SL_{d+1}(A) \cap E_{d+2}(A) \neq E_{d+1}(A)$. This means that the natural map $SL_{d+1}(A)/E_{d+1}(A) \longrightarrow SK_1(A)$ is *not* injective for such rings A.

In this section we show that if A is a regular affine algebra over a perfect C_1-field then

$$\frac{SL_{d+1}(A)}{E_{d+1}(A)} \longrightarrow SK_1(A)$$

is an isomorphism, where $d =$ Krull dimension of A.

We start with a result of Suslin, slightly modified to suit our needs. Namely, we use the observation by P. Raman that one may bypass Suslin's hypothesis that the ring is an affine algebra of an *irreducible* affine variety that is *nonsingular in codimension 1*.