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Hopf structure on the Van Est spectral sequence in K-theory

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In this paper, we study the van Est spectral sequence and its close relationship to K-theory and cyclic homology. The bicommutative Hopf algebra structure on the Van Est spectral sequence induces a long exact sequence of indecomposables. This leads us to Diagram C below and a proof of Karoubi's conjecture on the duality relationship of multiplicative K-theory and smooth group cohomology in some restricted cases. In the last section we reinterpret the Van Est spectral sequence as the Serre spectral sequence of a fibration of simplicial spaces. This paper is a sequel to [Ti] to which the reader is refered for further motivation.

I would like to thank the organizers for the opportunity to present these results at the conference.

1. Results.

Here we describe briefly the main results. The reader familiar with the earlier paper will see easily how Diagram C below is an improvement on Diagram B in [Ti] as now all rows are exact. The unfamiliar reader might find it helpful to read Sections 3 and 4 first which we hope are of interest in their own right and where we review many definitions in more detail.

Let A be a Banach or Fréchet algebra over $\mathbb{R}$, $GLA = \lim_{n\to\infty} GL_nA$ be the general linear group over A, and $\mathfrak{gl}A = \lim_{n\to\infty} \mathfrak{gl}_nA$ be its Lie algebra. If M_1 and M_2 are two infinite matrices over A, denote by $M_1 \oplus M_2$ the infinite matrix that acts on even coordinates like M_1 and on odd coordinates like M_2. This sum operation defines a map of groups $\oplus : GLA \times GLA \longrightarrow GLA$ and also of Lie algebras $\oplus : \mathfrak{gl}A \times \mathfrak{gl}A \longrightarrow \mathfrak{gl}A$. It is well known that over a field (here to be taken $\mathbb{R}$ or $\mathbb{C}$) this product and the diagonal map give the structure of a bicommutative Hopf algebra to the homology H_*GLA of the underlying topological space, to the Lie algebra homology $H_*^{Lie}\mathfrak{gl}A$, and to the group homology H_*BGLA_δ of the discrete group GLA_δ. The product is commutative as $M_1 \oplus M_2$ and $M_2 \oplus M_1$ only differ by conjugation via a permutation matrix. (See [L] for a proof in the case of $H_*^{Lie}\mathfrak{gl}A$).

In order to study the Van Est spectral sequence, we are more interested in cohomology, i.e. de Rham cohomology H^*GLA, continuous Lie algebra cohomology $H^*_{Lie}\mathfrak{gl}A$, and smooth group cohomology $H^*_{sm}GLA$. We will assume that *these are of finite type.* This ensures that $\oplus$ induces a well-defined comultiplication on these three algebras. For the Van Est spectral sequence we will also have to assume that *the de Rham complex of differential forms is split.* (See Section 4.) Let GLA_0 denote the identity component of GLA. The invariant differential forms on GLA_0 can be identified with the exterior algebra on $\mathfrak{gl}A$, and all three cohomology groups associated with GLA_0 are connected, that is H^0 is one dimensional.

PROPOSITION 1. *The van Est spectral sequence for GLA_0 is a spectral sequence of connected, bicommutative Hopf algebras with*

$$E_2 = H^*_{sm}GLA_0 \otimes H^*GLA_0 \quad \Longrightarrow \quad H^*_{Lie}\mathfrak{gl}A.$$

PROOF: The direct sum operation $\oplus : GLA_0 \times GLA_0 \to GLA_0$ is a group homomorphism and its induced map on the Lie algebra $\mathfrak{gl}A$ is again $\oplus$ as defined above, for $\exp(M_1 \oplus M_2) = \exp M_1 \oplus \exp M_2$. The proposition then follows by naturallity of the van Est spectral sequence. (See Section 4 and [Be] for more details.) $\diamondsuit$

COROLLARY 2. *There is an exact sequence of indecomposables*

$$\ldots \longrightarrow Q(H^*_{sm}GLA_0) \longrightarrow Q(H^*_{Lie}\mathfrak{gl}A) \longrightarrow Q(H^*GLA_0) \longrightarrow \ldots$$

PROOF: This follow from an application of Theorem 3.1 to the Van Est spectral sequence of Propostion 1. $\diamondsuit$

Using the notation Q for indecomposables and P for primitives as in Section 3, these vector spaces may now be identified as follows. By [LQ] or [T], the space of indecomposables $Q(H^*_{Lie}\mathfrak{gl}A)$ is isomorphic to the continuous cyclic cohomology groups $HC_c^{*-1}A$. For simplicity, we assume now that GLA *is connected.* Then, the cohomology H^*GLA is the dual Hopf algebra of H_*GLA and $Q(H^*GLA) = [P(H_*GLA)]^*$. Furthermore, as GLA is an associative H-space, rationally $P(H_*(GLA;\mathbb{Z}))$ is isomorphic to the homotopy groups $\pi_*GLA = K^{top}_{*+1}A$. Thus, $Q(H^*GLA)$ is essentially the dual of topological K-theory and we may use the suggestive notation $K^n_{top}A := [P(H_{n-1}GLA)]^*$. Similarly, we define $K^n_{alg}A := [P(H_nBGLA_\delta)]^*$ and $K^n_{rel} := [P(H_nGLA/GLA_\delta)]^*$. Hence, by passing to indecomposables, Diagram B in [Ti] may now be replaced by the commutative

DIAGRAM C.

$$\begin{tikzcd}
HC^n_cA \arrow[r, "I"] & HH^n_cA \arrow[r, "B"] \arrow[d, "D_{sm}"'] & HC^{n-1}_cA \arrow[r, "S"] \arrow[d, equal] & HC^{n+1}_cA \\
K^n_{top}A \arrow[r] \arrow[d, equal] & Q(H^n_{sm}GLA) \arrow[r] \arrow[d, "h"'] & HC^{n-1}_cA \arrow[r] \arrow[d, "ch_{rel}"'] & K^{n+1}_{top}A \arrow[d, equal] \\
K^n_{top}A \arrow[r] & K^n_{alg}A \arrow[r] & K^n_{rel}A \arrow[r] & K^{n+1}_{top}A
\end{tikzcd}$$

Here the top row is Connes' exact sequence for continuous cyclic and Hochschild cohomology. The middle row is that of Corollary 2 reinterpreted. The bottom row is also exact and is by definition the dual of the exact sequence that relates relative, Quillen's algebraic, and periodic K-theory. The vertical maps are described as follows. D_{sm} composed with h is the dual of the Dennis trace map, and ch_{rel} is the dual of Karoubi's relative Chern character.[1]

PROOF: If indeed we had just passed to indecomposables, there would be nothing to prove as all maps would be well defined and the commutativity of Diagram C would follow from that of Diagram B in [Ti]. However, in order to stay closer to to the K-homology groups, $K^*_{alg}A$ and $K^*_{rel}A$ have not been defined in terms of indecomposables. We thus need to thus define h and ch_{rel}. This can be done as follows.

[1]The missing vertical lines may be filled in by the dual of Karoubi's topological Chern character. See [Ti, §5] for further comments.

In [Ti] we constructed a factorization

$$HH_c^n A \xrightarrow{D_{sm}} H_{sm}^n GLA \longrightarrow H^n BGLA_\delta$$

of the dual Dennis trace map. Composing D_{sm} with the projection onto indecomposables gives the map $HH_c^n A \to Q(H_{sm}^n GLA)$. Now, in a bicommutative Hopf algebra, the natural map from the primitives to the indecomposables is an isomorphism. Thus we may think of the indecomposables as the subspace $P(H_{sm}^n GLA)$ to get a well defined map to $H^n BGLA_\delta$. Its image is contained in $P(H^\circ)$ where H° denotes the continuous dual Hopf algebra of $H = H_* BGLA_\delta$, i.e. H° is the largest Hopf algebra contained in $H^* BGLA_\delta$. But for all Hopf algebras we have $P(H^\circ) = [Q(H)]^*$ by a theorem of Michaelis [Mi]. Hence,

$$Q(H_{sm}^n GLA) = P(H_{sm}^n GLA) \xrightarrow{h} P(H^\circ) = [Q(H)]^* = [P(H)]^* = K_{alg}^n A$$

is well defined. Similarly, the factorization of the dual relative Chern character $HC_c^{n-1} A \to H_{Lie}^n \mathfrak{gl} A \to H^n GLA/GLA_\delta$ gives rise to the map $ch_{rel} : HC_c^{n-1} A \to K_{rel}^n A$. $\diamondsuit$

In [K] Karoubi also defines multiplicative K-groups $MK_n A$ such that they fit into a long exact sequence

$$\ldots \longrightarrow HC_{n-1}^c A \longrightarrow MK_n A \longrightarrow K_n^{top} A \longrightarrow \ldots$$

The middle row of Diagram C is just the dual of this sequence. This gives us a partial solution to a conjecture by Karoubi that the continuous dual of the multiplicative K-theory is the smooth group cohomology. We illustrate this with an example in the next section for $A = \mathbb{C}$.

2. Example $A = \mathbb{C}$.

We consider $\mathbb{C}$ as an algebra over $\mathbb{R}$. $GL\mathbb{C}$ is connected and it is well-known that its de Rham cohomology is an exterior algebra with one generator in each odd dimension:

$$H^*(GL\mathbb{C}, \mathbb{C}) = E_{\mathbb{C}}^*(x_1, x_3, \ldots, x_p, \ldots)$$