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Astérisque, tome 226 (1994), p. 435-452
<http://www.numdam.org/item?id=AST_1994__226__435_0>

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0. INTRODUCTION

It is a well known fact that the current Lie algebra $\mathcal{G}\otimes\mathbb{C}[[t,t^{-1}]]$ associated to a simple finite-dimensional Lie $\mathbb{C}$-algebra $\mathcal{G}$ has a central extension leading to the affine non-twisted Kac-Moody algebra $\mathcal{G}\otimes\mathbb{C}[[t,t^{-1}]] \oplus \mathbb{C}z$ with bracket

$$\{x\otimes f, y\otimes g\} = [x,y]\otimes fg + (x,y)Res\frac{df}{dt}g\ z$$

where $(\cdot\ ,\cdot)$ is the Killing form on $\mathcal{G}$ (cf. [Kac]).

In view of the known relationship between central extensions and the second (co)homology group with the coefficients in the trivial module, one of the main results of this paper can be considered as a generalization of this fact for general current Lie algebras, i.e. Lie algebras of the form $L\otimes A$ where L is a Lie algebra and A is associative commutative algebra, equipped with bracket

$$[x\otimes a, y\otimes b] = [x,y]\otimes ab.$$

Theorem 0.1. *Let L be an arbitrary Lie algebra over a field K of characteristic $p\neq 2$ and A an associative commutative algebra with unit over K. Then there is an isomorphism of K-vector spaces:*

$$\begin{aligned}(0.1)\quad H_2(L\otimes A) \ \simeq\ & H_2(L)\otimes A \ \oplus\ B(L)\otimes HC_1(A)\\ & \oplus\ \wedge^2(L/[L,L])\otimes Ker(S^2(A)\to A)\ \oplus\ S^2(L/[L,L])\otimes T(A)\end{aligned}$$

where the mapping $S^2(A)\to A$ induced by multiplication in A and $T(A) = \langle ab\wedge c + ca\wedge b + bc\wedge a | a,b,c\in A\rangle$.

Here $B(L)$ is the space of coinvariants of the L-action on $S^2(L)$, $HC_1(A)$ is the first-order cyclic homology group of A, and $\wedge^2$ and S^2 denote skew and symmetric products respectively. Notice that in the case $L=[L,L]$, the third and fourth terms in the right-hand side of (0.1) vanish.

Many particular cases of this theorem were proved by different authors previously. An exhaustive description of all previous works on this theme may be found in [H] and [S].

The first time a cohomology formula of the type (0.1) has appeared in [S] , where Theorem 0.1 was proved assuming that L is 1-generated over an augmentation ideal of its enveloping algebra. A. Haddi [H] obtained a result similar to Theorem 0.1 in the case when K is a field of characteristic zero (however, it seems that his arguments work over any field of characteristic $p \neq 2, 3$).

Our method of proof differs from all previous ones and is based on the Hopf formula expressing $H_2(L)$ in terms of a presentation $0 \to I \to \mathcal{L}(X) \to L \to 0$, where $\mathcal{L} = \mathcal{L}(X)$ is the free Lie algebra over K freely generated by the set X:

$$H_2(L) \simeq ([\mathcal{L}, \mathcal{L}] \cap I)/[\mathcal{L}, I] \tag{0.2}$$

(see, for example, [KS]).

The contents of the paper are as follows. §1 is devoted to some technical preliminary results. In §2 we determine the presentation of a current Lie algebra $L \otimes A$. In §3 Theorem 0.1 is proved. As it corollary we get in §4 a description of the space $B(L \otimes A)$. In §5 a "noncommutative version" of Theorem 0.1 is proved (Theorem 5.1). Namely, we derive the formula for the second homology group of a Lie algebra $(A \otimes B)^{(-)}$, where A, B are associative (noncommutative) algebras with unit, and $(-)$ in superscript denotes passing to an associated Lie algebra. The technique used here is no longer based on the Hopf formula, but on more or less direct computations in some factorspaces of cycles. However, arguments used in proof, resemble, to a great extent, the previous ones. Getting a particular case $B = M_n(K)$, we recover, after a slight modification, an isomorphism $H_2(sl_n(A)) \simeq HC_1(A)$ obtained in [KL].

The following notational convention will be used: the letters $a, b, c, \dots$, possibly with sub- and superscripts, denote elements of algebra A, while letters $u, v, w, \dots$ denote elements of the free Lie algebra $\mathcal{L}(X)$ with the set of generators $X = \{x_i\}$, if the otherwise is not stated. $\mathcal{L}^n(X)$ denotes the nth term in the derived series of $\mathcal{L}(X)$. The arrows $\rightarrowtail$ and $\twoheadrightarrow$ denote injection and surjection respectively.

All other undefined notions and notations are standard, and may be found, for example in [F] for Lie algebra (co)homology, and in [LQ] for cyclic homology. In some places we use diagram chasing and 3×3-Lemma without explicit to mention it.

ACKNOWLEDGEMENTS

This work was started during my graduate studies in the Institute of Mathematics and Mechanics of the Kazakh Academy of Sciences and finished in the Bar-Ilan University. My thanks due to Askar Dzhumadildaev and Steven Shnider, my former and present supervisors, for various sort of help, mathematical and other, and for interest in my work, and specially to Steven Shnider for careful reading of preliminary versions of manuscript. Also the financial support of Bar-Ilan University and Bat-Sheva Rotshild foundation is gratefully acknowledged. Finally, I am grateful to the organizers of the Conference on K-Theory held in June-July 1992 in Strasbourg for invitation and opportunity to present there the results of this paper.

1. PRELIMINARIES

Looking on the formula (0.1), one can distinguish between the first two "principal" terms and other two "non-principal" ones. In order to simplify calculations, we will obtain a variant of Hopf formula leading to the appearance of "principal" terms only, and then the general case will be derived.

Each nonperfect Lie algebra L, i.e. not coinciding with its commutant $[L, L]$, possess a "trivial" homology classes of 2-cycles with coefficients in the module K, namely, classes whose representatives do not lie in $L \wedge [L, L]$. More precisely, consider a natural homomorphism $\psi : H_2(L) \to H_2(L/[L, L]) \simeq \wedge^2(L/[L, L])$ and denote $H_2^{ess}(L) = Ker\psi$, the homology classes of "essential" cycles.

Lemma 1.1. *One has an exact sequence*

$$0 \to H_2^{ess}(L) \to H_2(L) \xrightarrow{\psi} \wedge^2(L/[L, L]) \xrightarrow{\pi} [L, L]/[[L, L], L] \to 0$$

where π is induced by multiplication in L.

Proof. This is just an obvious consequence of a 5-term exact sequence derived from the Hochschild-Serre spectral sequence $H_n(L/[L, L], H_m([L, L])) \Rightarrow H_{n+m}(L)$. ■

Further, we need a version of Hopf formula for $H_2^{ess}(L)$.

Lemma 1.2. *Given a presentation $0 \to I \to \mathcal{L} \to L \to 0$ of a Lie algebra L, one has*

$$H_2^{ess}(L) \simeq \frac{\mathcal{L}^3 \cap I}{\mathcal{L}^3 \cap [\mathcal{L}, I]}. \quad (1.1)$$

Proof. Since $L/[L,L] \simeq \mathcal{L}/(\mathcal{L}^2 + I)$, the Hopf formula (0.2) being applied to the algebra $L/[L,L]$ gives $H_2(L/[L,L]) \simeq \mathcal{L}^2/[\mathcal{L}, \mathcal{L}^2 + I]$, and

$$Ker\psi = Ker\left(\frac{\mathcal{L}^2 \cap I}{[\mathcal{L}, I]} \to \frac{\mathcal{L}^2}{[\mathcal{L}, \mathcal{L}^2 + I]}\right) \simeq \frac{\mathcal{L}^2 \cap I \cap [\mathcal{L}, \mathcal{L}^2 + I]}{[\mathcal{L}, I]} \simeq \frac{\mathcal{L}^3 \cap I}{\mathcal{L}^3 \cap [\mathcal{L}, I]}.$$

■

Now consider an action of a Lie algebra L on $S^2(L)$ via

$$[z, x \vee y] = [z, x] \vee y + x \vee [z, y].$$

Let $B(L) = S^2(L)/[L, S^2(L)]$ be the space of coinvariants of this action. The dual $B(L)^*$ is the space of symmetric bilinear invariant forms on L.

Let I, J be ideals of L. Define $B(I, J)$ to be the space of coinvariants of action of L on $I \vee J$. One has a natural embedding $B(I, J) \to B(L)$. The natural map $L \vee J \to (L/I) \vee ((I + J)/I)$ defines a surjection $B(L, J) \to B(L/I, (I + J)/I)$.

Lemma 1.3. *The short sequence*

$$0 \to B(L, I \cap J) + B(I, J) \to B(L, J) \to B(L/I, (I + J)/I) \to 0 \tag{1.2}$$

is exact.

Proof. Since $Ker(L \vee J \to L/I \vee (I + J)/I) = L \vee (I \cap J) + I \vee J$, the factorization through $[L, S^2(L)]$ yields

$$\begin{aligned} &Ker(B(L, J) \to B(L/I, (I + J)/I)) \\ &= (L \vee (I \cap J) + I \vee J + [L, S^2(L)])/[L, S^2(L)] \simeq B(L, I \cap J) + B(I, J) \end{aligned}$$

■

Remark. Actually we need the two following cases of this Lemma:

(1) $J = [L, L]$. Since $I \vee [L, L]$ and $[I, L] \vee L$ are congruent modulo $[L, S^2(L)]$ and $[I, L] \subseteq I \cap [L, L]$, then $B(I, [L, L]) \subseteq B(L, I \cap [L, L])$ and we get a short exact sequence

$$0 \to B(L, I \cap [L, L]) \to B(L, [L, L]) \to B(L/I, [L/I, L/I]) \to 0 \tag{1.3}$$

(2) $I = [L, L]$ and $J = L$. Then taking into account that for an abelian Lie algebra M, $B(M) \simeq S^2(M)$, the short exact sequence (1.2) becomes

$$0 \to B(L, [L, L]) \to B(L) \to S^2(L/[L, L]) \to 0 \tag{1.4}$$