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CYCLIC HOMOLOGY AND MODULI SPACES OF RIEMANN SURFACES

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1 Introduction

Let $\vec{\mathfrak{M}}(g)$ denote the moduli space of directed Riemann surfaces of genus g. It consists of conformal equivalence classes of triples $[F, O, X]$, where F is a closed Riemann surface and X is a tangent direction at some point O of F. Since the mapping class group $\vec{\Gamma}(g) = \pi_0(\mathit{Diff}^+(F, O, X))$ acts freely on the contractible Teichmüller space $\vec{\mathfrak{T}}(g)$ of marked directed surfaces, the quotient $\vec{\mathfrak{M}}(g)$ is an orientable, open manifold of dimension $6g - 3$ with the homotopy type of $B\vec{\Gamma}(g)$. The group $\vec{\Gamma}(g)$ is better known as the mapping class group of genus g surfaces with one boundary curve.

This moduli space $\vec{\mathfrak{M}}(g)$ can be described as a configuration space $\mathfrak{P}(g)$ of slits in the complex plane; we recall this uniformization from [Bö 1]. A compactification $P(g)$ was developed in [Bö 2]. It has a cell structure whose cellular chain complex resembles formally the Hochschild resolution of an non-commutative algebra without unit; and in addition, there is a cyclic operation and an involution on the set of cells.

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This analogy is strong enough to permit the definition of Hochschild homology groups $HH_*(P(g))$ and cyclic homology groups $HC_*(P(g))$ for these complexes. They are related to their (topological) homology groups $H_*(P(g))$ by long exact sequences

$$\cdots \longrightarrow HHN_*(P(g)) \longrightarrow HH_*(P(g)) \longrightarrow H_{*-1}(P(g)) \longrightarrow \cdots$$

and

$$\cdots \longrightarrow HH_*(P(g)) \longrightarrow HC_*(P(g)) \longrightarrow HC_{*-2}(P(g)) \longrightarrow \cdots$$

in which $HHN_*(P(g))$ is the so-called naive Hochschild homology. Our intention is to use the apparatus of cyclic homology theory to study the spaces $P(g)/W(g)$, which are Poincaré dual to the moduli spaces $\vec{\mathfrak{M}}(g)$. Here we merely report on some basic ideas.

We point out that $\vec{\mathfrak{M}}(g)$ carries a (non-free) $\mathbb{S}^1$-action given by rotation of the tangent vector or X. The quotient is the moduli space of genus g surfaces with one puncture. It seems difficult to describe this action on the homeomorphic space $\mathfrak{P}(g)$; but we expect this action to be related to the cyclic action on cells. Complex conjugation of conformal structures is another symmetry on $\vec{\mathfrak{M}}(g)$; it is easily seen to transform to the reflection operator mentioned above.

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2 Moduli and parallel slit domains

We recall a specific description of the moduli space $\vec{\mathfrak{M}}(g)$; the reader is refered to [Bö 1] for more details.

Let $[F, O, X] \in \vec{\mathfrak{M}}(g)$ be a directed Riemann surface of genus g. There is a function $u : F \longrightarrow \bar{\mathbb{R}} = \mathbb{R} \cup \infty$ with the following properties: (1) u is harmonic away from O, and (2) $u(z) - Re(1/z)$ is smooth and vanishes at O for any local parameter z defined around O such that $z(O) = 0$ and $dz(x)$ is positive-proportional to $-dx$, where x is any non-zero tangent vector at O representing the direction X. Such a function exists by the standard existence theorems for differentials on Riemann surfaces; it is unique up to a positive real factor and up to a real additive constant.

Let the critical graph K of the gradient flow of u consist of the dipole O and all zeroes of the flow as vertices, and all integral curves which leave zeroes are the edges. Since $F_0 = F \setminus K$ is connected and simply-connected, the restriction of u to F_0 is the real part of a holomorphic map $w = u + i\,v : F_0 \longrightarrow \mathbb{C}$; w is unique, up to another additive constant for the harmonic-conjugate v of u. The complement of $w(F_0) \subseteq \mathbb{C}$, — which can be described as a configuration of $2g$ pairs of horizontal slits in the complex plane $\mathbb{C}$, — will comprise all moduli of the conformal class $[F, O, X]$.

A slit L_k is a horizontal half-line, which starts at some point $z_k = (x_k, y_k) \in \mathbb{C}$, and is unbounded to the left. There are always $4g$ slits, paired by a fixed point free involution λ in the symmetric group $\mathfrak{S}_{4g}$, acting on the index set $\mathbf{I} = \{1, \ldots, 4g\}$. A configuration is subject to two conditions:

$$y_k \leq y_{k+1} \tag{1}$$

$$x_k = x_{\lambda(k)} \tag{2}$$

So far no assumption is made about the slits being disjoint or different.

To associate a surface $F(L)$ to L we glue, for each pair k and $\lambda(k)$, the upper (resp. lower) bank of L_k to the lower (resp. upper) bank of $L_{\lambda(k)}$. As basepoint we choose $O = \infty$ and X is the direction of $-dx$ under the local parameter $\zeta \mapsto 1/\zeta$. If $F(L)$ is a (non-singular) surface, it inherits from $\mathbb{C}$ a conformal structure, and thus $[F, O, X] \in \vec{\mathfrak{M}}(g)$. In case $F(L)$ has singularities, or if it is a surface of a genus smaller than g, we call L degenerate.

The following conditions (3) and (4) guarantee that neither O nor any finite point of $F(L)$ is singular and that $F(L)$ has maximal genus g. Define a new permutation $\sigma = \lambda \circ t$, where t denotes the cyclic rotation $k \mapsto k + 1 \ (mod\ 4g)$. Let $\kappa(\lambda) + 1$ denote the number of cycles of σ, which can be any even number between 0 and $2g$. We admit only pairings λ for which

$$\kappa(\lambda) = 0 \tag{3}$$

holds; such a λ is called connected.

The next condition excludes certain subconfigurations.

(4) *There is no index k such that :*

$$\lambda(k) = k+2, \quad L_k = L_{k+2} \quad \textit{and} \quad L_{k+1} \subseteq L_k.$$

In [Bö 2] we examined in detail, what type of singular surfaces occur if (3) or (4) is violated.

It is obvious from the gluing process that two distinct configurations can lead to conformally equivalent surfaces. In this case they are connected by a chain of moves (called Rauzy-moves) of the following type: if $L_{k-1} \subseteq L_k$ then L_{k-1} can jump to the upper bank of the slit $L_{\lambda(k)}$. In its new position it will be contained in $L_{\lambda(k)}$, and all slits overtaken by this move change their index by a cyclic rotation, and λ is conjugated accordingly. Such a move leaves $F(L)$ certainly invariant. The equivalence classes generated by Rauzy moves are denoted by $\mathfrak{L} = [L_1, \ldots, L_{4g}|\lambda]$. A class is called non-degenerate, if none of its representatives violates (3) or (4). In the older literatur such a class is called a parallel slit domains.

On the space of all parallel slit domains the contractible 3-dimensional group of similarities of $\mathbb{C}$ acts freely as a group of conformal invariants. It is generated by translations in the x- and y-direction and by dilations; the parameters of such a transformation correspond precisely to the three undetermined constants in the complex potential w. We therefore introduce the following normalizations.

$$y_1 = 0 \tag{5}$$
$$y_{4g} = 1 \tag{6}$$
$$min\{x_k\} = 0 \tag{7}$$

These conditions are invariant under moves, and thus conditions on a class. For a non-degenerate class we always have $y_1 < y_{4g}$, enabling us to normalize as in (5) and (6). The main result of [Bö 1] is that the space of all non-degenerate, normalized configuration classes is homeomorphic to the moduli space $\vec{\mathfrak{M}}(g)$.

It will be convenient for the compactification to introduce the additional condition

$$max\{x_k\} < 1. \tag{8}$$