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Introduction

Topological cyclic homology associates to a ring R a spectrum $\mathrm{TC}(R)$. The homotopy groups $\mathrm{TC}_i(R)$ are connected to Connes' cyclic homology groups of R, but they are stronger invariants than the cyclic homology groups. There is a map, called the cyclotomic trace, from Quillen's K-theory spectrum $K(R)$ into $\mathrm{TC}(R)$. This map is conjectured to be a p-homotopy equivalence for certain complete semi local rings, and in particular for rings of integers in local fields with positive residue characteristic p. We refer the reader to [25] for further discussion of the cyclotomic trace. In this paper we set up a general strategy for calculating topological cyclic homology, and we apply it to the key case where the ring in question is the ring of p-adic integers.

Let us very briefly describe the construction of topological Hochschild and cyclic homology. If we in the standard simplicial Hochschild complex of R, whose homotopy groups are the Hochschild homology groups, replace the ring with the Eilenberg-MacLane spectrum it determines, and the tensor product (over $\mathbb{Z}$) with smash product of spectra, then we obtain the topological Hochschild homology spectrum $\mathrm{THH}(R)$. There are severe technical difficulties in carrying out the indicated substitutions, but they were overcome in [8] by the introduction of functors with smash product. A ring R gives in particular rise to such a functor. The resulting spectrum $\mathrm{THH}(R)$ turns out to be an equivariant S^1-spectrum with deloops in the direction of every representation. Following [24] one would then expect that the topological cyclic homology to be closely related to the homotopy orbit spectra $EC_{n+} \wedge_{C_n} \mathrm{THH}(R)$. This is indeed the case, but instead of taking homotopy quotients with respect to the finite cyclic groups it is better to take fixed sets $\mathrm{THH}(R)^{C_n}$. The fixed sets contain many strata, one for each subgroup of C_n with the homotopy quotient above corresponding to the free strata. The spectrum $\mathrm{TC}(R)$ is a certain homotopy inverse limit of the fixed sets over a category which contains the inclusions of fixed sets and certain maps which mix the strata, cf. [11], [22] and sect.1 below.

The content of the paper is as follows. In the first section we introduce the concept of a p-cyclotomic spectrum. It is an equivariant S^1-spectrum with some extra structure,

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and THH (R) belongs to this class of spectra for any functor with smash product and any prime p. The extra structure consists of S^1-maps

$$\varphi_{C_p} : \mathrm{THH}\,(R)^{C_p} \to \mathrm{THH}\,(R)$$

different from the inclusion of fixed sets, where C_p is the cyclic group of order p. For a p-cyclotomic spectrum T there are the important cofibrations

$$EC_{p^n+} \wedge_{C_{p^n}} T \to T^{C_{p^n}} \xrightarrow{\Phi} T^{C_{p^{n-1}}} \tag{0.1}$$

with Φ equal to the $C_{p^{n-1}}$-fixed set of φ_{C_p}. The sphere spectrum is a cyclotomic spectrum, and the above cofibrations are split for $T = S^0$, giving the standard decomposition from [34] of the fixed point spectra.

The second section of the paper discusses the norm cofibration

$$EG_+ \wedge_G T \to \mathrm{Map}_G(EG_+, T) \to \hat{\mathbb{H}}(G;T) \tag{0.2}$$

for an equivariant G-spectrum T. Special cases of this fibration have been considered by various authors. We need for our purpose the Greenlees-May foundations from [19]. There are spectral sequences for each of the terms of (0.2). For example one has

$$\hat{H}^*(G;\pi_*T) \Rightarrow \pi_*\hat{\mathbb{H}}(G;T) \tag{0.3}$$

where $\hat{H}^*$ denotes the Tate cohomology. The main result in sect. 2 is Theorem 2.15, which relates the differential structure of (0.3) to the exact homotopy sequences of (0.2).

In section three we tabulate the spectral sequence

$$H^*(C_{p^n};\pi_*J) \Rightarrow \pi_*\mathrm{Map}\,(EC_{p^n+}, J) \tag{0.4}$$

where J is the periodic image of J space, and where we use homotopy groups with $\mathbb{F}_p$ coefficients for an odd prime p.

The next two sections four and five discuss the structure of (0.3) when $T = \mathrm{THH}\,(\mathbb{Z}_p)$ and $G = C_{p^n}$. The natural map from $S^0 = \mathrm{THH}\,(*)$ to $\mathrm{THH}\,(\mathbb{Z}_p)$ is trivial on homotopy groups, but induces a highly non-trivial map

$$\mathrm{Map}_G\left(EC_{p^n+}, S^0\right) \to \mathrm{Map}_{C_{p^n}}(EC_{p^n+}, \mathrm{THH}\,(\mathbb{Z}_p)) \tag{0.5}$$

It just changes filtration. The E^2-term $H^*\left(C_{p^{n-1}}, \pi_*J\right)$ of (0.4) is a direct summand in theE^2-term of the domain of (0.5), and injects into theE^2-term of the range. Due to the filtration shift there are problems however in proving that the E^r-terms of (0.4) injects into the E^r-term of the range in (0.5). Assuming this to be true, however, the structure of (0.4) implies the structure of

(0.6) $$H^*(C_{p^n}; \pi_*\mathrm{THH}(\mathbb{Z}_p)) \Rightarrow \pi_*\mathrm{Map}_{C_{p^n}}(EC_{p^n+}, \mathrm{THH}(\mathbb{Z}_p))$$

for all n. This is our Conjecture 4.3. In section five it is shown that Conjecture 4.3 is indeed true if the unit map from S^0 to $K(\mathbb{Z}_p)$ factors over the J-spectrum. Such a factorization is known to exist on the level of the 0-th spaces of the spectra by [28]. There are other possible attacks on Conjecture 4.3, than to prove the factorization. The most promising is to use that the unit maps into the cyclic 1-skeleton of $\mathrm{THH}(\mathbb{Z}_p)$ when composed with the trace map, but at the time of writing we have not been able to carry this to a definite conclusion.

The rest of the paper is based upon Conjecture 4.3, at least in part. Section six and section seven compare (0.1) and (0.2) and show that (0.1) is homotopy equivalent to the 0-connected cover of (0.2) via the obvious maps which inject fixed sets into homotopy fixed sets. This uses Conjecture 4.3 for general n. For $n = 1$ and $n = 2$, however, we can prove the conjecture, and for these values of n, (0.1) and the (-1)-connected cover of (0.2) do agree. Combining the results of section two and section four one then obtains the homotopy groups with $\mathbb{F}_p$ coefficients of $\mathrm{TC}(\mathbb{Z}_p)$, p odd. Section eight proves periodicity: multiplication with v_1 induces an isomorphism between $\pi_r(\mathrm{TC}(\mathbb{Z}_p); \mathbb{F}_p)$ and $\pi_{r+2(p-1)}(\mathrm{TC}(\mathbb{Z}_p); \mathbb{F}_p)$ for $r \geq 0$. In section nine we use the linearization map from $\mathrm{TC}(*)$ to $\mathrm{TC}(\mathbb{Z}_p)$, the known structure of $\mathrm{TC}(*)$, from [11], and a theorem of J. Rognes [33], to show that

(0.7) $$\mathrm{TC}(\mathbb{Z}_p)^\wedge_p \simeq (\mathrm{Im}\, J \times \mathbb{Z})^\wedge_p \times B(\mathrm{Im}\, J \times \mathbb{Z}_p)^\wedge_p \times (\Sigma bu)^\wedge_p$$

This result, however is dependent on Conjecture 4.3.

We note that (0.7) is the expected structure of $K(\mathbb{Z}_p)^\wedge_p$ according to the generalized Lichtenbaum-Quillen conjecture as formulated by Dwyer and Friedlander. It lends credit to the belief that the cyclotomic trace is a homotopy equivalence, after p-completion, for these kind of rings.

The final section ten has the character of an appendix. Its main result shows that relative K-theory is mapped monomorphically to the relative topological Hochschild homology in the first non-trivial dimensions. As a consequence we derive an unpublished result of the first named author, which is used in a critical way in section six.

The scheme set up here for evaluating $\mathrm{TC}(-)$ has been applied to a number of other rings. For example, one knows by now $\mathrm{TC}(R)^\wedge_p$ for $R = \mathbb{F}$, $\mathbb{F}[[t]]$, $\mathbb{F}[t, t^{-1}]$ and $\mathbb{F}_p[\varepsilon]/(\varepsilon^2)$, cf. [22] and [25]. For example, $\mathrm{TC}(\mathbb{F}) \simeq H\mathbb{Z}_p$, the Eilenberg-MacLane spectrum of $\mathbb{Z}_p$, when $\mathbb{F}$ is a finite field with p^a elements. Furthermore the only non-zero homotopy groups of TC $(\mathbb{F}[\epsilon]/(\epsilon^2))$ are:

$$p \text{ odd: } \mathrm{TC}_{2n-1}\Big(\mathbb{F}[\epsilon]/\big(\epsilon^2\big)\Big) = \widehat{W}_{2n-1}(\mathbb{F})^{\langle -1\rangle} \text{ and } \mathrm{TC}_0\Big(\mathbb{F}[\epsilon]/\big(\epsilon^2\big)\Big) = \mathbb{Z}_p$$

$$p = 2: \quad \mathrm{TC}_{2n-1}\Big(\mathbb{F}[\epsilon]/\big(\epsilon^2\big)\Big) = \mathbb{F}^{\oplus n} \text{ and } \mathrm{TC}_0\Big(\mathbb{F}[\epsilon]/\big(\epsilon^2\big)\Big) = \mathbb{Z}_2$$

Here $\widehat{W}_n(\mathbb{F})$ denotes the Witt-vectors of length n, i.e.

$$\widehat{W}_n(\mathbb{F}) = (1 + X\mathbb{F}[[X]])^*/\Big(1 + X^{n+1}\mathbb{F}[[X]]\Big)^*,$$

and the superscript $< -1 >$ indicates the -1 eigenspace for the involution on $\widehat{W}(\mathbb{F})$ which changes sign on X. This predicts then the values for the "tangent space of $K(\mathbb{F})$".

Further advancement in the understanding of TC (R) is dependent upon a more thorough understanding of THH (R) than is available at present. We refer the reader to the discussion given in [25]. Apart from the obvious unsolved problem of proving Conjecture 4.3, the present paper raises at least two other issues, namely to calculate TC $(\mathbb{Z}_2)$ and to calculate TC (A) for rings of integers in local fields of positive residue characteristic. Ideally, one might hope to describe TC (A) for a Galois extension $A/\mathbb{Z}_p$ as a functor TC $(\mathbb{Z}_p)$ and the extension.

The calculations performed in sections six to eight are somewhat unpleasant, and not well understood from an algebraic point of view. One feels that some algebraic notion could be developed to explain and streamline them. In particular one would like to have a good description of the structure of the homotopy groups of the homotopy S^1-orbit of THH $(\mathbb{Z}_p)$. This will be needed for example in the calculation of $\mathrm{TC}(\mathbb{Z}_p[C_{p^n}])$.

The present paper has been a long time under way, and several people have contributed with very helpful comments. We in particular want to acknowledge the help we have had from L. Hesselholt and J. Rognes. J. Rognes read a draft of the entire paper, corrected several mistakes, and gave many valuable suggestions for improvements in the exposition. Finally we owe to him the characterization of the spectrum $\Sigma bu_p^\wedge$ used in section nine.

Added in January 1994. It appears that Stavros Tsalidis in his 1994 Ph.D thesis from Brown University has proved Conjecture 4.3 below, so that (0.7) and the other conditional results of this paper are in fact theorems. See Remark 6.9 below for a little more details.

Combined with a second recent result due to R. McCarthy that the diagram

$$\begin{array}{ccc} K(R)_p^\wedge & \to & \mathrm{TC}(R)_p^\wedge \\ \downarrow & & \downarrow \\ K(R/I)_p^\wedge & \to & \mathrm{TC}(R/I)_p^\wedge \end{array}$$