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HOMOTOPY OF PROJECTIONS IN C^*-ALGEBRAS OF STABLE RANK ONE

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1. DISCUSSION AND INTRODUCTION

My talk at the Orleans conference concerned extremally rich C^*-algebras, based on joint work with G. K. Pedersen. A unital C^*-algebra A is called *extremally rich* if its closed unit ball, A_1, is the convex hull of its extreme points, and a non-unital C^*-algebra A is *extremally rich* if and only if $\widetilde{A}$, the result of adjoining an identity, is extremally rich. For other equivalent definitions and most of the properties the reader is referred to our forthcoming papers [4] and [5]. The theory of extremal richness is not involved in the first theorem of this paper, but some of it will be discussed in this section, mainly for heuristic purposes and partly as a research announcement.

A theorem of Rørdam [15] implies that every C^*-algebra of stable rank one is extremally rich. In fact $tsr(A) = 1$ if and only if A is extremally rich and every extreme point of $\widetilde{A}_1$ is unitary. (The second condition is a kind of finiteness condition - see further discussion in §3. Of course every unitary is extremal.) There is an intermediate concept, isometric richness, and A is isometrically rich if and only if it is extremally rich and every extreme point of $\widetilde{A}_1$ is an isometry or co-isometry. Thus for prime C^*-algebras, in particular for primitive or simple C^*-algebras, isometric richness is equivalent to extremal richness. Rørdam [16, 4.5] and Pedersen [12, 10.1] state (without the terminology) that every purely infinite simple C^*-algebra is extremally rich. In fact a simple C^*-algebra is extremally rich if and only if it is either purely infinite or of stable rank one.

A guiding philosophy is that if a property of C^*-algebras A is known both when A is purely infinite simple and when $tsr(A) = 1$, then one can hope to prove it for A extremally rich. This hope is substantially realized with respect to the invariance properties of extremal richness (strong Morita equivalence, passage to hereditary subalgebras, behavior under extensions) and with respect to certain non-stable K-theoretic properties, but this hope is very possibly wrong with respect to other non-stable K-theoretic properties. Moreover, the facts stated in the abstract show that it is wrong with respect to the question of Zhang stated there, though there is a way to re-formulate the question (see §3). Actually the issue of how close is the relationship between extremal richness and non-stable K-theory is very much up in the air. Conceivably all of the results already proved could be special cases of more

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general results that do not mention extremal richness, but this seems unlikely to me at present.

A (possibly non-unital) C^*-algebra A is said to have *weak cancellation* if whenever p and q are projections in $A \otimes \mathcal{K}$ such that each of p, q generates the same (closed, two-sided) ideal J and p and q have the same image in $K_0(J)$, then $p \sim q$. Here $\mathcal{K}$ is the C^*-algebra of compact operators on a separable infinite-dimensional Hilbert space and $\sim$ denotes Murray–von Neumann equivalence. Of course, C^*-algebras of stable rank one satisfy the strong cancellation property that if p and q have the same image in $K_0(A)$, then $p \sim q$ (Blackadar [1, 6.5.1]). (I am here not stating the cancellation property in quite the usual way, but the discrepancy is only technical and not important when $tsr(A) = 1$.) A related fact is that $tsr(A) = 1$ implies that the natural map from $K_0(J)$ to $K_0(A)$ is injective. This is a known result (folklore) for which I do not know a reference, and the corresponding statement for extremally rich C^*-algebras is false. Cuntz [7] showed that purely infinite simple C^*-algebras have weak cancellation.

A unital C^*-algebra A is said to have *K_1-surjectivity* if the natural map from $U(A)/U_0(A)$ to $K_1(A)$ is surjective, *K_1-injectivity* if this map is injective, and *K_1-bijectivity* if it is bijective. Here $U(A)$ is the unitary group of A and $U_0(A)$ the connected component of the identity. Cuntz [7] showed that purely infinite simple C^*-algebras have K_1-bijectivity (the same is true of $\widetilde{A}$ if A is non-unital), and Rieffel [14] showed that $tsr(A) = 1$ implies K_1-bijectivity.

It will be shown in [5] that every isometrically rich C^*-algebra has weak cancellation and also every extremally rich C^*-algebra of real rank zero. There are other positive theorems, but it is not known whether every extremally rich C^*-algebra has weak cancellation. Every extremally rich C^*-algebra with weak cancellation also has K_1-surjectivity and certain other properties. It was shown by Lin [10] that every C^*-algebra of real rank zero has K_1-injectivity. There is presently no real evidence of a relationship between K_1-injectivity and extremal richness.

Zhang [21] formulated the question of the abstract as follows. For a projection p in A let $\mathcal{G}_p = \{upu^{-1} : u \in U(\widetilde{A})\}$. The question is whether $\mathcal{G}_p$ is connected. Note that a projection q is in $\mathcal{G}_p$ if and only if $q \sim p$ and $1 - q \sim 1 - p$. Thus if $tsr(A) = 1$, q is in $\mathcal{G}_p$ if and only if $q \sim p$. Also the results of Cuntz [7] show that if A is purely infinite simple, then q is in $\mathcal{G}_p$ if and only if $q \sim p$ and either $p = q = 1$ or $p \neq 1$ and $q \neq 1$. It is well known that if p and q are homotopic projections in A (i.e., p and q are in the same connected component of the set of projections of A), then $q = upu^{-1}$ for some u in $U_0(A)$ or $U_0(\widetilde{A})$. All of this must be kept in mind since the results in the literature prior to [21] are not formulated in the same way.

2. First Theorem and example

The proof of the theorem has much in common with the proof of the quoted result of Effros and Kaminker [8, 2.4]. A different proof, perhaps more suggestive, is sketched in §3.

Theorem 1. *If p is a projection in a C^*-algebra A and if $tsr(A) = 1$, then $\mathcal{G}_p$ is connected. In other words, unitarily equivalent projections in A are homotopic.*

Proof. Without loss of generality we may assume A is unital. Let I be the ideal of A generated by p, and let $B = I + \mathbb{C} \cdot 1$. Then $tsr(B) = 1$ by results of Rieffel [13]. Let q be an element of $\mathcal{G}_p$. Since $q \sim p$ in A, q is in I and $q \sim p$ in I. Thus q is unitarily equivalent to p in B (as indicated above, because of the cancellation property for B). Let u be in $U(B)$ such that $q = upu^{-1}$, and let α be the image of u in $K_1(B)$. Because pIp is a full hereditary C^*-subalgebra of I the natural map from $K_1(pIp)$ to $K_1(I)$ is a bijection (see the following remark). Also $K_1(B) = K_1(\widetilde{I}) = K_1(I)$. Thus α is the image of an element β of $K_1(pIp)$. Now $tsr(pIp) = 1$ (see the following remark) and hence β is the image of a unitary v_1 in $U(pIp)$ by a result of Rieffel [14] (K_1-surjectivity). Let $v = v_1 + 1 - p$ and $w = uv^{-1}$. Then w is in $U(B)$, $q = wpw^{-1}$, and the image of w in $K_1(B)$ is 0. Thus the K_1-injectivity result of [14] implies w is in $U_0(B)$. This clearly implies that q is homotopic to p.

Remark. I have not been able to find a reference for the fact that inclusions of full hereditary C^*-algebras induce isomorphisms of K-groups in full generality. Paschke [11, 1.2] proves this when both algebras are σ-unital. The general case can be deduced from this by using direct limits of separable subalgebras.

I also do not know a reference for the fact that $tsr(A) = 1$ implies $tsr(B) = 1$ when B is a hereditary C^*-subalgebra of A, though surely this fact is known. If I is the ideal of A generated by B, then $tsr(I) = 1$ by Rieffel [13] and I is strongly Morita equivalent to B. Rieffel also proves in [13] that the property $tsr = 1$ is preserved by stable isomorphism, which is the same as strong Morita equivalence when both algebras are σ-unital. A proof that the property $tsr = 1$ is preserved by strong Morita equivalence in general will be found in [4], and also a direct (and short) proof that $tsr = 1$ passes to hereditary subalgebras can be obtained from [4].

EXAMPLE. Let $\{e_n : n \in \mathbb{Z}\}$ be an orthonormal basis for a Hilbert space H. Let U in $B(H)$ be the bilateral shift defined by $Ue_n = e_{n+1}$, let P in $B(H)$ be the projection on $\overline{\{e_n : n \geq 0\}}$, and let P_0 be the projection on $\overline{\{e_n : n > 0\}}$. Let $A = \{T \in B(H) : PT(1-P), (1-P)TP \in \mathcal{K}\}$. Then A is a C^*-algebra, $U, P, P_0 \in A$, $UPU^{-1} = P_0$, and P is not homotopic to P_0 in A.

Only the last assertion needs proof. Note that $A \supset \mathcal{K}$ and $A/\mathcal{K} \cong \mathcal{Q} \oplus \mathcal{Q}$, where $\mathcal{Q}$ is the Calkin algebra. Thus every element V of $U_0(A)$ has an image $v_1 \oplus v_2$ in $\mathcal{Q} \oplus \mathcal{Q}$ and $v_1, v_2 \in U_0(\mathcal{Q})$. It follows that PVP and $(1-P)V(1-P)$ have index 0 as operators on PH and $(1-P)H$. In particular, $U \notin U_0(A)$. But if P and P_0 were homotopic, there would be V in $U_0(A)$ such that $VPV^{-1} = P_0$, and thus $V = UW$ with $WP = PW$. Since the last equation implies $W \in U_0(A)$, we have a contradiction.

Now A has real rank zero but is not extremally rich. That $RR(A) = 0$ follows from the facts that $RR(\mathcal{K}) = RR(A/\mathcal{K}) = 0$ and projections lift. That A is not extremally rich follows from [4] and the fact that there is an extremal element u in $A/\mathcal{K}$ which is not the image of an extremal element of A. Namely, $u = v \oplus v^*$ where v is a proper isometry in $\mathcal{Q}$. To get the promised example, we replace A with a C^*-subalgebra B such that U and P, and hence P_0, are in B. Then P and P_0 are still unitarily equivalent projections in B which are not homotopic.

Let D in $B(H)$ be defined by $De_n = \lambda^n e_n$, where λ is a complex number of absolute value 1 which is not a root of unity. Note that D is unitary and $DP = PD$. Let

$$B_1 = C^*(PD, PUP) \text{ and } B_2 = C^*((1-P)D, (1-P)U(1-P)) .$$

Then $B_1 \supset \mathcal{K}(PH)$ and $B_2 \supset \mathcal{K}((1-P)H)$, since PUP and $(1-P)U(1-P)$ are respectively a unilateral shift and a backward shift. Since $DU = \lambda UD$, it is easy to see that $B_1/\mathcal{K}(PH) \cong C^*(u_1, d_1)$ and $B_2/\mathcal{K}(1-P)H) \cong C^*(u_2, d_2)$, where u_i and d_i are unitary and $d_i u_i = \lambda u_i d_i$. Then $C^*(u_1, d_1)$ and $C^*(u_2, d_2)$ are isomorphic to the same irrational rotation algebra C. That $tsr(C) = 1$ and $RR(C) = 0$ follows *a fortiori* either from Blackadar, Kumjian, and Rørdam [2] or from Elliott and Evans [9]. (This subject has a distinguished history, and various combinations of earlier papers could have been used for constructing an example.) Now let $B = \{T \in A : PTP \in B_1 \text{ and } (1-P)T(1-P) \in B_2\}$. Thus $\mathcal{K} \subset B$ and $B/\mathcal{K} \cong C \oplus C$. Since $RR(\mathcal{K}) = RR(B/\mathcal{K}) = 0$, to show that $RR(B) = 0$ we need only show that every projection in $B/\mathcal{K}$ is the image of a projection in B (see [3, 3.14], [18, 2.4], or [20, 2.3]). The fact that projections always lift when the ideal is $\mathcal{K}$ follows from a well known result of Calkin [6]. It will be shown in [4] that whenever a C^*-algebra B has an ideal which is a dual C^*-algebra such that the quotient algebra is isometrically rich, then B is extremally rich. This applies to the algebra called B here. Note also that B is primitive, so that B is even isometrically rich.

3. More discussion and second theorem

Let p be a fixed projection in a unital C^*-algebra A. Proposition 3.2 of Zhang [21] implies $\mathcal{G}_p$ is connected if and only if the natural map from

$$U(pAp)/U_0(pAp) \oplus U((1-p)A(1-p))/U_0((1-p)A(1-p))$$

to $U(A)/U_0(A)$ is surjective.

Now let I and J be the ideals of A generated by p and q, respectively, and let $s(I, J)$ be the natural map from $K_1(I) \oplus K_1(J)$ to $K_1(A)$. Then it is clear from the previous paragraph that a necessary condition for connectedness of $\mathcal{G}_p$ is that certain elements of $K_1(A)$, namely those in the image of $U(A)/U_0(A)$, be in the range of $s(I, J)$. If A has K_1-surjectivity, it is necessary that $s(I, J)$ be surjective. Note that $s(I, J)$ is surjective if and only if the natural map from $K_0(I \cap J)$ to $K_0(I) \oplus K_0(J)$ is injective and that this is so if $tsr(A) = 1$ (or merely if either $tsr(I) = 1$ or $tsr(J) = 1$). If $s(I, J)$ is surjective and if both pAp and $(1-p)A(1-p)$ have K_1-surjectivity, then for any u in $U(A)$ there is a v such that $vp = pv$ and uv^{-1} is in the kernel of the map from $U(A)/U_0(A)$ to $K_1(A)$. (Here we use the facts that $K_1(pAp) \cong K_1(I)$, $K_1((1-p)A(1-p)) \cong K_1(J)$, cf. proof of Theorem 1.) Then if also A has K_1-injectivity, we see that $\mathcal{G}_p$ is connected.

Thus if we make enough hypotheses about K_1-surjectivity and –injectivity, then $\mathcal{G}_p$ is connected if and only if $s(I, J)$ is surjective. Of course, the example in the previous section was deliberately constructed so that $s(I, J)$ would not be surjective. Now the surjectivity of $s(I, J)$ strikes me as an interesting condition, but it may be desirable