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OPERATOR ALGEBRAS, FREE GROUPS AND OTHER GROUPS

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1. *INTRODUCTION.*

Let Γ be a group. We denote by $\mathbb{C}[\Gamma]$ the group algebra of complex linear combinations of elements of Γ, given together with the involution

$$X = \sum_{\gamma\in\Gamma} z_\gamma \gamma \mapsto X^* = \sum_{\gamma\in\Gamma} \overline{z_\gamma}\gamma^{-1} .$$

The **operator algebras** of interest here are various completions of $\mathbb{C}[\Gamma]$. Non abelian **free groups** are among the most studied examples of groups in this context. We denote by F_n the non abelian free group on n generators, where n is either an integer, $n \geq 2$, or $n = \infty$, meaning an infinite countable number of generators.

Our guiding principle is that the special case of free groups indicates typical behaviours which hold in many other cases of geometrical interest. This has suggested the three main aspects of the report below :

a survey of some properties of operator algebras associated to the F_n 's,

an exploration of "geometric" groups giving rise to algebras with similar properties,

a list of open problems (some of them are numbered, from 1 to 19, and others appear in the text).

We shall concentrate on groups Γ which are lattices in semi-simple Lie groups ([Rag], [Mas]) or hyperbolic [Gr1], and on algebras which are either von Neumann algebras or C^*-algebras. But we shall mention on occasions other groups and other algebras. Unless explicitely stated otherwise, Γ denotes a **countable** group and operator algebras are **separable** in the appropriate sense.

Many important developments are left untouched. In particular, we say very little on K-theory and KK-theory related to group C^*-algebras, and nothing at all on the Novikov conjecture.

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2. *THE VON NEUMANN ALGEBRA* $W^*_\lambda(\Gamma)$.

2.1. *Generalities.*

For a Hilbert space $\mathcal{H}$, we denote by $\mathcal{L}(\mathcal{H})$ the involutive algebra of bounded operators on $\mathcal{H}$ and by $\mathcal{U}(\mathcal{H})$ the group of unitary operators on $\mathcal{H}$. Any **unitary representation** $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ of a group Γ gives rise to a morphism of involutive algebras $\mathbb{C}[\Gamma] \to \mathcal{L}(\mathcal{H})$ which is again denoted by π, and defined by

$$\pi\left(\sum_{\gamma\in\Gamma} z_\gamma \gamma\right) = \sum_{\gamma\in\Gamma} z_\gamma \pi(\gamma) .$$

We denote by $W^*_\pi(\Gamma)$ the weak closure of $\pi(\mathbb{C}[\Gamma])$ in $\mathcal{L}(\mathcal{H})$.

Consider in particular the space $l^2(\Gamma)$ of square summable complex valued functions on Γ and the left regular representation

$$\lambda : \Gamma \to \mathcal{U}\left(l^2(\Gamma)\right)$$

where $(\lambda(\gamma)\xi)(x) = \xi(\gamma^{-1}x)$ for all $\gamma, x \in \Gamma$ and for all $\xi \in l^2(\Gamma)$. The weak closure $W^*_\lambda(\Gamma)$ of $\lambda(\mathbb{C}[\Gamma])$ is the **von Neumann algebra** of Γ.

There is a **finite normal trace** $\tau : W^*_\lambda(\Gamma) \to \mathbb{C}$ which extends the map $\mathbb{C}[\Gamma] \to \mathbb{C}$ given by $\sum_{\gamma\in\Gamma} z_\gamma\gamma \mapsto z_1$, and this trace is faithful. Thus the von Neumann algebra $W^*_\lambda(\Gamma)$ is finite, of the form $W_I \oplus W_{II} = (\oplus_{i=1}^\infty W_i) \oplus W_{II}$ with each W_i of type I_i, say with unit e_i, and with W_{II} of type II_1, say with unit e.

One has $e = 0$ if and only if Γ contains an abelian group of finite index. Let Γ_f denote the subgroup of Γ of elements with finite conjugacy classes and let $D\Gamma_f$ denote its commutator subgroup; then one has $e = 1$ if and only if either $[\Gamma : \Gamma_f] = \infty$ or $[\Gamma : \Gamma_f] < \infty$ and $|D\Gamma_f| = \infty$. See [Kan], [Sm1] and [Tho]. In case Γ is finitely generated, one has either $e = 0$ or $e = 1$. (This appears in [Ka2], but it is also a straightforward consequence of [Kan]. Indeed $e \neq 1$ implies $[\Gamma : \Gamma_f] < \infty$ by [Kan, Satz 1]; as Γ_f is also finitely generated in this case, the centre of Γ_f is of finite index in Γ_f [Tom, Corollary 1.5], and thus also in Γ; consequently $e = 0$.) But there are already in [Kap] examples, due to B.H. Neumann, which show that one may have $0 \neq e \neq 1$. Here is one of these examples : for each $i \in \mathbb{N}$, denote by D_i a copy of the dihedral group of order 8 and by C_i its center, which is of order 2 and which is also its derived group; let B be the direct sum of the B_i 's and let C be the subgroup of elements $(c_i)_{i\in\mathbb{N}} \in B$ such that $c_i \in C_i$ for each $i \in \mathbb{N}$ and $\prod_{i\in\mathbb{N}} c_i = 1$; this B.H. Neumann example is the quotient A/B; its von Neumann algebra is the direct product of $\mathbb{C}$ (with $\tau(e_1) = 1/2$) and of a factor of type II_1 (with $\tau(e) = 1/2$).

For a group Γ, Kaplansky has observed that $\tau(e_1)$ is the inverse of the order of the derived group of Γ [Kap, Theorem 1]. There are formulas giving $1 - e$ [Fo2]. The sum $1 - e = \sum e_i$ is finite, and indeed $e_i = 0$ whenever $i^2 > |\Gamma/\Gamma_f|$ [Sm2]. When $e \neq 1$, one has $W_I \approx W^*_\lambda(\Gamma/\Gamma_0)$, where the von Neumann kernel Γ_0 of Γ is defined as $\bigcap_\pi Ker(\pi : \Gamma \to U(n))$, the intersection being over all finite dimensional representations of Γ [Sch, Satz 1].

Let Γ be a group such that $0 \neq e \neq 1$; I do not know whether there exists a group Γ_{II} naturally associated to Γ and such that $W_{II} \approx W_{\lambda}^{*}(\Gamma_{II})$. Here is a similar question : let Γ be a group such that $W_{\lambda}^{*}(\Gamma)$ is *not* a II_1-factor but contains a central projection c such that $cW_{\lambda}^{*}(\Gamma)$ is a II_1-factor; does there exist a group Γ_c naturally associated to Γ and such that $cW_{\lambda}^{*}(\Gamma) \approx W_{\lambda}^{*}(\Gamma_c)$?

Observe that, in case Γ is a hyperbolic group, Γ_f is precisely the so-called *virtual center* of Γ, denoted by $Z_{virt}(\Gamma)$ in [Cha].

2.2. *Free groups.*

Historically, the first examples of factors of type II_1 are given by Murray and von Neumann in [MNI], as crossed products which involve abelian groups (indeed subgroups of $\mathbb{R}$) acting ergodically on appropriate spaces. Several years later, they give a new construction which is "considerably simpler than our previous procedures, but it is clearly related to them" [MNIV, Introduction, §5]. Among other things, they show the following results. Recall that a group Γ has **infinite conjugacy classes,** or in short is icc, if *all* its conjugacy classes distinct from $\{1\}$ are infinite; for example, F_n is icc for all $n \geq 2$.

Theorem 1 (Murray and von Neumann).
(i) Let Γ be a group. Then $W_{\lambda}^{*}(\Gamma)$ is a factor if and only if Γ is icc.
(ii) For each $n \geq 2$ the factor $W_{\lambda}^{*}(F_n)$ does not possess Property Gamma.

This is shown in [MNIV] : see Lemma 5.3.4 for (i), Definition 6.1.1 for Property Gamma and §6.2 for (ii) when $n = 2$; moreover Lemma 6.3.1 shows that $W_{\lambda}^{*}(\Gamma_1 \star \Gamma_2)$ is a factor which does not possess Property Gamma whenever Γ_1 [respectively Γ_2] is a group containing at least two [resp. three] elements (the star denotes a free product). About the meaning of (ii), let us recall that a von Neumann algebra M does *not* have Property Gamma if and only if it is **full**, namely if and only if the group $Int(M)$ of its inner automorphisms is closed in the group $Aut(M)$ of all its automorphisms (see [Co74, Corollary 3.8] and [Co76, Theorem 2.1]).

Though we do *not* consider twisted crossed products in this report, let us at least mention that many of the results discussed here have "twisted formulations". For example, for claim (i) of Theorem 1 above, see [Pac, Proposition 1.3].

Claim (ii) suggests immediately the following, which is Problem 4.4.44 in [Sak].

Problem 1. *Does it happen that $W_{\lambda}^{*}(F_n) \approx W_{\lambda}^{*}(F_{n'})$ for $n \not\approx n'$?*

Though Problem 1 is still open, progress has been obtained recently, using Voiculescu's theory of freeness in noncommutative probability spaces (see among others [Vo2], [VDN] and [Sk2]). For example, one must have

either $W_{\lambda}^{*}(F_n) \approx W_{\lambda}^{*}(F_{n'})$ for all n, n' such that $2 \leq n, n' \leq \infty$
or $W_{\lambda}^{*}(F_n) \not\approx W_{\lambda}^{*}(F_{n'})$ for all n, n' such that $2 \leq n < n' \leq \infty$.

This has been first proved for $n, n' < \infty$, independently by K. Dykema and F. Radulescu; moreover, this holds for $n, n' \leq \infty$ by [Ra5, Corollary 4.7]. Let us also mention that

$$W^*_\lambda(\star_{n=1}^\infty \Gamma_n) \approx W^*_\lambda(F_\infty)$$

whenever Γ_n is a nontrivial amenable group for all $n \geq 1$ (see [Vo2, Corollary 3.5] and [Dy2, Corollary 5.4]), and that

$$W^*_\lambda(\Gamma \star \Gamma') \approx W^*_\lambda(F_2)$$

when Γ , Γ' are infinite amenable groups [Dy2, particular case of Corollary 5.3]. See also [HaVo].

One of the novelties connected with the results above is the discovery, due independently to K. Dykema [Dy1] and F. Radulescu [Ra4], of a continuous family of II_1-factors $L(F_r)$ interpolating the free group factors. In the next theorem, we denote by $M_1 \star M_2$ the II_1-factor which is the reduced free product of two finite factors M_1, M_2; this is a crucial notion in Voiculescu's approach [Vo2].

Theorem 2 (Dykema, Radulescu, Voiculescu). *For each extended real number r such that $1 < r \leq \infty$, there exists a II_1-factor $L(F_r)$ such that*

$L(F_r) \star L(F_{r'}) \approx L(F_{r+r'})$ *for all* $r, r' \in]1, \infty]$,
$p(L(F_r) \otimes M_n(\mathbb{C})) \approx L(F_{1+\gamma^{-2}(r-1)})$ *for any* $r \in]1,\infty]$ *and any projection* $p \in L(F_r) \otimes M_n(\mathbb{C})$ *of trace* $\gamma \in]0, \infty[$ *(where n is large enough),*
$L(F_n) \approx W^*_\lambda(F_n)$ *for all* $n \in \{2, 3, ..., \infty\}$,
$L(F_r) \otimes M \approx L(F_{r'}) \otimes M$ *for all* $r, r' \in]1, \infty[$ *whenever M is either* $\mathcal{L}(\mathcal{H})$, *or* R, *or* $W^*_\lambda(F_\infty)$,
the isomorphism class of $L(F_r) \otimes L(F_{r'})$ *depends only on* $(r-1)(r'-1)$, *for all* $r, r' \in]1, \infty]$.

In the theorem, $\mathcal{L}(\mathcal{H})$ denotes the factor of type I_∞ and R denotes the hyperfinite factor of type II_1; moreover p is of trace γ for the trace of value 1 on the unity of $L(F_r)$. The first result quoted after Problem 1 is in fact

either $L(F_r) \approx L(F_{r'})$ for all r, r' such that $1 < r, r' \leq \infty$

or $L(F_r) \not\approx L(F_{r'})$ for all r, r' such that $1 < r < r' \leq \infty$.

Let us mention that some attention has been paid to free groups on uncountably many generators : if $F_{!!}$ denotes such a free group, then $W^*_\lambda(F_{!!})$ hasn't any "regular MASA"; also (we anticipate here on Section 3) the reduced C^*-algebra $C^*_\lambda(F_{!!})$, which clearly is not separable, has only separable abelian *-subalgebras [Po1, Section 6].

2.3. *Other groups.*

Considerable effort has been devoted to understand whether various factors of the form $W^*_\lambda(\Gamma)$ are or are not isomorphic to each other. The oldest result of this kind follows from Claim (ii) of Theorem 1 above on one hand and from the consideration of locally finite groups which are icc on the other hand; this result, which is the existence of two non isomorphic factors of type II_1, is recorded as the achievement of Chapters V and VI in [MNIV, Theorem XVI]. Later, the same construction $\Gamma \mapsto W^*_\lambda(\Gamma)$ has