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Plan

Introduction

In this paper, we study *operator spaces* in the sense of the theory developed recently by Blecher-Paulsen [BP] and Effros-Ruan [ER1]. By an operator space, we mean a closed subspace $E \subset B(H)$, with H Hilbert. In the category of operator spaces, the morphisms are the *completely bounded maps* for which we refer the reader to [Pa1]. Let $E \subset B(H)$, $F \subset B(K)$ be operator spaces (H, K Hilbert). A map $u\colon E \to F$ is called completely bounded ($c.b.$ in short) if

$$\sup_{n\geq 1} \|I_{M_n} \otimes u\|_{M_n(E)\to M_n(F)} < \infty$$

where $M_n(E)$ and $M_n(F)$ are equipped with the norms induced by $B(\ell_2^n(H))$ and $B(\ell_2^n(K))$ respectively. We denote

$$\|u\|_{cb} = \sup_{n\geq 1} \|I_{M_n} \otimes u\|_{M_n(E)\to M_n(F)}.$$

The map u is called a complete isomorphism if it is an isomorphism and if u and u^{-1} are $c.b.$. We say that $u\colon E \to F$ is a complete isometry if for each $n \geq 1$ the map $I_{M_n} \otimes u\colon M_n(E) \to M_n(F)$ is an isometry. We refer to [Ru, ER2-7, B1, B2] for more information on the rapidly developing Theory of Operator Spaces.

We will be mainly concerned here with the "geometry" of *finite dimensional* operator spaces. In the Banach space category, it is well known that every separable space embeds isometrically into ℓ_∞. Moreover, if E is a finite dimensional normed space then for each $\varepsilon > 0$, there is an integer n and a subspace $F \subset \ell_\infty^n$ which is $(1+\varepsilon)$-isomorphic to E, i.e. there is an isomorphism $u\colon E \to F$ such that $\|u\|\,\|u^{-1}\| \leq 1+\varepsilon$.

Here of course, n depends on ε, say $n = n(\varepsilon)$ and usually (for instance if $E = \ell_2^k$) we have $n(\varepsilon) \to \infty$ when $\varepsilon \to 0$.

Quite interestingly, it turns out that this fact is not valid in the category of operator spaces: although every operator space embeds completely isometrically into $B(H)$ (the non-commutative analogue of ℓ_∞) it is not true that a finite dimensional operator space must be close to a subspace of M_n (the non-commutative analogue of ℓ_∞^n) for some n. The main object of this paper is to study this phenomenon.

We will see that this phenomenon is very closely related to the remarkable work of E. Kirchberg on exact C^*-algebras. We will show that some of Kirchberg's ideas can be developed in a purely "operator space" setting. Our main result in the first section is Theorem 1, which can be stated as follows.

Let $B = B(\ell_2)$ and let $K \subset B$ be the ideal of all the compact operators on ℓ_2.

If X, Y are operator spaces, we denote by $X \otimes_{\min} Y$ their minimal (or spatial) tensor product. If $X \subset B(H)$ and $Y \subset B(K)$, this is just the completion of the linear tensor product $X \otimes Y$ for the norm induced by $B(H \otimes_2 K)$.

Let $\lambda \geq 1$ be a fixed constant.

The following properties of an operator space X are equivalent:

(i) The sequence

$$\{0\} \to K \otimes_{\min} X \to B \otimes_{\min} X \to (B/K) \otimes_{\min} X \to \{0\}$$

is exact and the map

$$T_X\colon\ (B \otimes_{\min} X)/(K \otimes_{\min} X) \to (B/K) \otimes_{\min} X$$

has an inverse T_X^{-1} with norm $\|T_X^{-1}\| \leq \lambda$.

(ii) for each $\epsilon > 0$ and each finite dimensional subspace $E \subset X$, there is an integer n and a subspace $F \subset M_n$ such that $d_{cb}(E, F) < \lambda + \epsilon$.

Here $d_{cb}(E, F)$ denotes the $c.b.$ analogue of the Banach-Mazur distance (see (0) below for a precise definition.) We will denote by $d_{SK}(E)$ the infimum of $d_{cb}(E, F)$ when F runs over all operator spaces F which are subspaces of M_k for some integer k.

One of the main results in section 2 can be stated as follows (see Theorem 7 below).

Consider $F \subset M_k$ with $\dim F = n$ and $k \geq n$ arbitrary, then for any linear isomorphism $u\colon\ \ell_\infty^n \to F^*$ we have

$$\|u\|_{cb}\|u^{-1}\|_{cb} \geq n[2(n-1)^{1/2}]^{-1}.$$

In particular this is > 1 for any $n \geq 3$. Here the space F^* is the dual of F with its "dual operator space structure" as explained in [BP, ER1, B1, B2].

Equivalently, if we denote by E_1^n the operator space dual of ℓ_∞^n, (this is denoted by $\max(\ell_1^n)$ in [BP]) then we have

$$d_{SK}(E_1^n) \geq \frac{n}{2\sqrt{n-1}}.$$

We also show a similar estimate for the space which is denoted by $R_n + C_n$ in [P1]. Moreover, we show that the n-dimensional operator Hilbert space OH_n (see [P1]) satisfies

$$d_{SK}(OH_n) \geq \left(\frac{n}{2\sqrt{n-1}}\right)^{1/2}.$$

These estimates are asymptotically sharp in the sense that $d_{SK}(E_1^n)$ and $d_{SK}(R_n+C_n)$ are $O(n^{1/2})$ and $d_{SK}(OH_n)$ is $O(n^{1/4})$ when n goes to infinity.

Later on in the paper, we show that the operator space analogue of the "Banach Mazur compactum" is not compact and we prove various estimates related to that phenomenon. (The noncompactness itself was known, at least to Kirchberg.) We will include several simple facts on ultraproducts of finite dimensional operator spaces which are closely connected to the discussion of "exact" operator spaces presented in section 1. Let us denote by OS_n the set of all n-dimensional operator spaces. We consider that two spaces E, F in OS_n are the same if they are completely isometric. Then the space OS_n is a metric space when equipped with the distance

$$\delta(E, F) = \log d_{cb}(E, F).$$

We include a proof that OS_n is complete but not compact (at least if $n \geq 3$) and we give various related estimates. As pointed out to me by Kirchberg, it seems to be an open problem whether OS_n is a *separable* metric space.[1]

In passing, we recall that in [P1] we proved that $d_{cb}(E, OH_n) \leq n^{1/2}$ for any E in OS_n and therefore that

$$\sup\{d_{cb}(E, F) \mid E, F \in OS_n\} = n.$$

Actually, that supremum is attained on the subset $HOS_n \subset OS_n$ formed of all the *Hilbertian* operator spaces (i.e. those which, as normed spaces, are isometric to the Euclidean space ℓ_2^n). We also show that (at least for $n \geq 3$) HOS_n is a closed but non compact subset of OS_n. Perhaps the subset HOS_n is not even separable.[1] In section 5, we show the following result. Let E be any operator space and let $C \geq 1$ be a constant. Fix an integer $k \geq 1$. Then there is a compact set T and a subspace $F \subset C(T) \otimes_{\min} M_k$ such that $d_{cb}(E, F) \leq C$ iff for any operator space X and any $u \colon X \to E$ we have

$$\|u\|_{cb} \leq C\|u\|_k,$$

where $\|u\|_k = \|u\|_{M_k(X) \to M_k(E)}$.

Notation: Let (E_m) be a sequence of operator spaces. We denote by $\ell_\infty\{E_m\}$ the direct sum in the sense of ℓ_∞ of the family (E_m). As a Banach space, this means that $\ell_\infty\{E_m\}$ is the set of all sequences $x = (x_m)$ with $x_m \in E_m$ for all m with $\sup_m \|x_m\|_{E_m} < \infty$ equipped with the norm $\|x\| = \sup_m \|x_m\|_{E_m}$. The operator space structure on $\ell_\infty\{E_m\}$ is defined by the identity

$$\forall n \quad M_n(\ell_\infty\{E_m\}) = \ell_\infty\{M_n(E_m)\}.$$

[1] See the Note added at the end of this paper.

Equivalently, if $E_m \subset B(H_m)$ (completely isometrically) then $\ell_\infty\{E_m\}$ embeds (completely isometrically) into $B(\oplus_m H_m)$ as block diagonal operators.

We will use several times the observation that if F is an other operator space then $\ell_\infty\{E_m\} \otimes_{\min} F$ embeds completely isometrically in the natural way into $\ell_\infty\{E_m \otimes_{\min} F\}$. In particular, if F is finite dimensional these spaces can be completely isometrically identified.

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§1. Exact operator spaces.

Let E, F be operator spaces. We denote

$$d_{cb}(E,F) = \inf\{\|u\|_{cb}\|u^{-1}\|_{cb}\} \tag{0}$$

where the infimum runs over all isomorphisms $u\colon E \to F$. If E, F are not completely isomorphic we set $d_{cb}(E,F) = \infty$. This is the operator space version of the Banach Mazur distance. We will study the smallest distance of an operator space E to a subspace of the space $K = K(\ell_2)$ of all compact operators on ℓ_2. More precisely, this is defined as follows

$$d_{SK}(E) = \inf\{d_{cb}(E,F) \mid F \subset K\}. \tag{1}$$

Let F be a finite dimensional subspace of K. By an entirely classical perturbation argument one can check that for each $\varepsilon > 0$ there is an integer n and a subspace $\widetilde{F} \subset M_n$ such that $d_{cb}(F, \widetilde{F}) < 1 + \varepsilon$. It follows that for any finite dimensional operator space E we have

$$d_{SK}(E) = \inf\{d_{cb}(E,F) \mid F \subset M_n, \quad n \geq 1\}. \tag{1$'$}$$

In his remarkable work on exact C^*-algebras (cf. [Ki]) Kirchberg introduces a quantity which he denotes by $locfin(E)$ for any operator space E. His definition uses completely positive unit preserving maps. The number $d_{SK}(E)$ appears as the natural "*c.b.*" analogue of Kirchberg's $locfin(E)$. Note that $d_{SK}(E)$ is clearly an invariant of the operator space E and we have obviously

$$d_{SK}(E) \leq d_{SK}(F) d_{cb}(E,F) \tag{1$''$}$$

for all operator spaces E, F.

Let X be an operator space. We will say that X is exact if the sequence

$$\{0\} \to K \otimes_{\min} X \to B \otimes_{\min} X \to (B/K) \otimes_{\min} X \to \{0\} \tag{2}$$

is exact. In other words, X is an exact operator space if the natural completely contractive map

$$B \otimes_{\min} X \to (B/K) \otimes_{\min} X$$