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by Sorin Popa

Dedicated to Professor Ciprian Foias, on his 60'th birthday

Introduction

We will show in this paper that, unlike central sequences (i.e., commuting-independent sequences) which in general may or may not exist, free-independent sequences exist in any separable type II_1 factor.

More generally, we will in fact prove the following:

Theorem. *Let $N \subset M_\infty$ be an inclusion of separable type II_1 factors. Assume there exists an increasing sequence of von Neumann subalgebras $N \subset M_n \subset M_\infty$ such that $\overline{\bigcup_n M_n} = M_\infty$ and such that $N' \cap M_n$ is finite dimensional for all n. Then there exists a unitary element $v = (v_n)_n$ in the ultrapower algebra N^ω ([D1]) such that*

$$M_\infty \vee vM_\infty v^* = M_\infty \underset{N'\cap M_\infty}{*} vM_\infty v^*.$$

Here $P_1 \underset{B}{*} P_2$ denotes the finite von Neumann algebra free product with amalgamation, with its free trace $\tau_1 * \tau_2$, where (P_1, τ_1), (P_2, τ_2) are finite von Neumann algebras with their corresponding finite, normal, faithful traces, and with $B \subset P_1$, $B \subset P_2$ a common subalgebra such that $\tau_{1|_B} = \tau_{2|_B}$ ([Po6], [V2]).

In the particular case when $N \subset M = M_\infty$ are factors and $N' \cap M = \mathbb{C}$, for example when $N = M$, the amalgamated free product is a genuine free product ([V1]) and any element of the form vxv^*, $x \in M$ is free with respect to M. Thus we get:

Corollary. *If $N \subset M$ is an inclusion of type II_1 factors with trivial relative commutant then there exist unitary elements $(u_n)_n$ in N that are free independent with respect to M, i.e., such that $\tau(u_n^k) = 0$, $\forall n$, $\forall k \neq 0$, and $\lim_{n\to\infty} \tau(u_n^{k_1} b_1 u_n^{k_2} b_2 \cdots u_n^{k_\ell} b_\ell) = 0$ for any $\ell \geq 1$ and any $b_1, \cdots, b_\ell \in M$, $\tau(b_i) = 0$, $1 \leq i \leq \ell - 1$, $k_1, k_2, \cdots, k_\ell \in \mathbb{Z} \setminus \{0\}$.*

Since the notion of "independent events" in classical probability theory becomes "free independence" in the noncommutative probability of ([V3]), our result on the existence of free-independent sequences can be regarded as the "free" analogue of the results on the existence of nontrivial central sequences in a factor ([D1], [McD], [C2]) or a subfactor ([Bi]). There are thus some notable differences between central and free-independent sequences: Nontrivial central sequences may not exist in general, but they always form an algebra while free sequences always exist, though the set of all such sequences doesn't form an algebra. Also, the existence of noncommuting central sequences in a factor M implies that M splits off the hyperfinite type II_1 factor, i.e., $M \simeq M \otimes R$, but, although all factors have free-independent sequences, neither the hyperfinite nor the property T factors ([C3]) are free products of algebras (cf. [MvN], [Po5]). Along these lines note also that, while taking the free product $M * R$ of a property T factor M by R cancels the property T for $M * R$, the fundamental group of $M * R$ will remain countable (cf. [Po5]), yet $M \otimes R$ will have fundamental group $\mathbb{R}_+^*$. Thus, as also pointed out in ([V1,3]), the analogy between tensor and free products seems, in certain respects, rather limited.

The above theorem was first stated, without a proof, in Sec. 8 of [Po6]. But in fact it was obtained prior to the rest of the results in [Po6]. It was this theorem that led us to the construction of irreducible subfactors of arbitrary index s, $N^s(Q) \subset M^s(Q)$, by using free traces on amalgamated free product algebras. Indeed, when suitably interpreted the theorem shows that such inclusions $N^s(Q) \subset M^s(Q)$ can be asymptotically recovered in any other irreducible inclusion of same index s.

The paper is organized as follows. In Sec. 1 we prove the technical results needed for the proof of the theorem. The proofs are inspired from (2.1 in [Po4]), where a slightly weaker version of the results here were obtained. The proofs rely on the local quantization principle ([Po1, 7]) and on a maximality argument, like in [Po4]. Conversely, the results in [Po1, 7] are immediate consequences of the theorem and its corollary, giving some sharp estimates as a bonus. This fact will be explained in Sec. 2, where the main result of the paper, a generalization of the above stated theorem, is proved, (see 2.1) and some more immediate corollaries are deduced. We expect it in fact to be useful for approaching some other problems as well, an aspect on which we comment in Sec. 3. Thus, we speculate on the possibility of having a functional analytical characterization of the free group algebras, on the indecomposability of such algebras and their possible embedding into other algebras. We also include a construction of separable type II_1 factors M with the fundamental group $\mathcal{F}(M)$ countable but containing any prescribed countable subset of $(0, \infty)$, e.g., with $\mathcal{F}(M) \supset \mathbb{Q}$.

We are grateful to D. Voiculescu for stimulating us to write down the proof of the result announced in Sec. 8 of [Po4], through his constant interest and motivating comments.

1 Some technical results

In what follows all finite von Neumann algebras are assumed given with a normal, finite, faithful trace, typically denoted by τ. For standard notations and terminology, we refer the reader to [Po6, 7].

We will also often use the following:

1.1. Notation. Let B be a von Neumann algebra. If $v \in B$ is a partial isometry with $v^*v = vv^*$, $S \subset B$ is a subset and $k \leq n$ are nonnegative integers then denote $S_v^{0,n} \stackrel{\text{def}}{=} S$ and $S_v^{k,n} \stackrel{\text{def}}{=} \left\{ b_0 \underset{i=1}{\overset{k}{\pi}} v_i b_i \mid b_i \in S,\ 1 \leq i \leq k-1,\quad b_0,\ b_k \in S \cup \{1\} \text{ and } \quad v_i \in \{v^j | 1 \leq |j| \leq n\} \right\}$.

The next lemma is the crucial technical result needed to prove the theorem in this paper:

1.2. Lemma. *Let $N \subset M$ be an inclusion of type II_1 von Neumann algebras. Assume $N' \cap M$ is finite dimensional. Let $\varepsilon > 0$, n a positive integer, $F \subset M$ a finite set and $f \in N$ a projection of scalar central trace in N such that $E_{N' \cap M}(b) = 0$, for all $b \in fFf$. Then there exists a partial isometry v in fNf such that:*

a) $v^*v = vv^*$ and its central trace in N is a scalar $> \dfrac{\tau(f)}{4}$.

b) $\|E_{N' \cap M}(x)\|_1 \leq \varepsilon, \quad x \in \underset{k=1}{\overset{n}{\cup}} F_v^{k,n}$.

Proof. Let $\delta > 0$. Denote $\varepsilon_0 = \delta$, $\varepsilon_k = 2^{k+1}\varepsilon_{k-1}$, $k \geq 1$. Let $\mathcal{W} = \{v \in fNf | v$ partial isometry, $v^*v = vv^*$, the central trace of v^*v in N is a scalar, and $\|E_{N' \cap M}(x)\|_1 \leq \varepsilon_k \tau(v^*v)$, for all $1 \leq k \leq n$, $x \in F_v^{k,n}\}$. Endow $\mathcal{W}$ with the order $\leq$ in which $v_1 \leq v_2$ iff $v_1 = v_2 v_1^* v_1$. $(\mathcal{W}, \leq)$ is then clearly inductively ordered. Let v be a maximal element in $\mathcal{W}$. Assume $\tau(v^*v) \leq \tau(f)/4$. If w is a partial isometry in pNp, where $p = f - v^*v$, and if $u = v + w$ then for $x = b_0 \underset{i=1}{\overset{k}{\pi}} u_i b_i \in F_u^{k,n}$ we have

$$(1) \qquad x = b_0 \prod_{i=1}^{k} v_i b_i + \sum_{\ell} \sum_{i} z_0^i \prod_{j=1}^{\ell} w_{i_j} z_j^i$$

where $k \geq \ell \geq 1$, $i = (i_1, \ldots, i_\ell)$ with $1 \leq i_1 < \cdots < i_\ell \leq k$, $w_{i_j} = w^s$ if $v_{i_j} = v^s$, $z_0^i = b_0 v_1 b_1 \cdots b_{i_1 - 1} p$, $z_j^i = p b_{i_j} v_{i_j+1} \cdots v_{i_{j+1}} p$, for $1 \leq j < \ell$ and $z_\ell^i = p b_{i_\ell} v_{i_\ell+1} \cdots v_k b_k$ and where the sum is taken over all $\ell = 1, 2, \cdots, k$ and all $i = (i_1, \ldots, i_\ell)$.

By (A.1.4 in [Po7]), given any $\alpha > 0$ there exists a projection q in pNp, of scalar central support in pNp (and thus in N), such that

$$(2) \qquad \|qzq - E_{(N'\cap M)p}(z)q\|_{1,pMp} < \alpha\tau_{pMp}(q)$$

for all z of the form z^i_j, for some $\ell \geq 2$, some $i = (i_1, \cdots, i_\ell)$ and $1 \leq j \leq \ell - 1$.

In the case $\ell = 2$ and $i_1 = 1$, $i_2 = k$, if we take the partial isometry $w \in pNp$ so that $w^*w = ww^* = q$, then we get for $z = pb_1v_2b_2\cdots v_{k-1}b_{k-1}p$:

$$\begin{aligned}
(3) \quad & \|E_{N'\cap M}(b_0w_1b_1v_2b_2\cdots v_{k-1}b_{k-1}w_kb_k)\|_1 \\
&\leq \|w_1b_1v_2b_2\cdots b_{k-1}w_k\|_1 \\
&= \|qb_1v_2b_2\cdots b_{k-1}q\|_1 \\
&= \|qzq\|_1 = \|qzq\|_{1,pMp}\tau(p) \\
&\leq \left(\|E_{(N'\cap M)p}(z)q\|_{1,pMp} + \alpha\tau_{pMp}(q)\right)\tau(p) \\
&= \left(\|E_{(N'\cap M)p}(z)\|_{1,pMp}\tau(q)/\tau(p) + \alpha\tau(q)/\tau(p)\right)\tau(p) \\
&= \left(\|E_{(N'\cap M)}(z)\|_1\tau(p)^{-1} + \alpha\right)\tau(q).
\end{aligned}$$

But since for $x \in N' \cap M$, v and $p = vv^*$ commute with x we get by taking into account that $vb_1v_2b_2\cdots v_{n-1}b_{n-1}v^* \in F_v^{k,n}$ and $b_1v_2b_2\cdots v_{k-1}b_{k-1} \in F_v^{k-2,n}$ the following estimate:

$$\begin{aligned}
(4) \quad & \|E_{N'\cap M}(z)\|_1 \\
&= \sup\{|\tau(zx)| \mid x \in N' \cap M, \ \|x\| \leq 1\} \\
&= \sup\{|\tau(pb_1v_2b_2\cdots v_{k-1}b_{k-1}x)| \mid x \in N' \cap M, \ \|x\| \leq 1\} \\
&\leq \sup\{|\tau(b_1v_2b_2\cdots v_{k-1}b_{k-1}x)| \mid x \in N' \cap M, \ \|x\| \leq 1\} \\
&\quad + \sup\{|\tau(vb_1v_2b_2\cdots v_{k-1}b_{k-1}v^*x)| \mid x \in N' \cap M, \ \|x\| \leq 1\} \\
&= \|E_{N'\cap M}(b_1v_2b_2\cdots v_{k-1}b_{k-1})\|_1 \\
&\quad + \|E_{N'\cap M}(vb_1v_2b_2\cdots b_{k-1}v^*)\|_1 \\
&\leq \varepsilon_{k-2}\tau(v^*v) + \varepsilon_k\tau(v^*v).
\end{aligned}$$

By combining (3) and (4) and noting that $\tau(v^*v) \leq \tau(f)/4$ implies $\tau(v^*v)/\tau(p) \leq \tau(f)/3$, it follows that if we take $\alpha \leq \delta/3 < (\varepsilon_k - \varepsilon_{k-2})/3$ then we get:

$$(5) \qquad \|E_{N'\cap M}(b_0w_1b_1v_2b_2\cdots v_{k-1}b_{k-1}w_kb_k)\|_1 \leq 2/3\varepsilon_k\tau(q).$$

Note now that $z = pb_1v_2b_2\cdots v_{k-1}b_{k-1}p$ is the only element of the form z^i_j for which $i_2 - i_1 = k - 1$ and that it appears in the sum (1) only once, in the writing of the