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1 Introduction

A notion of a Gel'fand pair for compact quantum groups introduced by T.H. Koornwinder in [19] is a generalization of the classical one for a locally compact group G and its compact subgroup K such that for any irreducible unitary representation of G, the dimension of the space of K-bi-invariant matrix elements is not greater then 1; this is equivalent to the commutativity of the subalgebra of group algebra of G, formed by K-bi-invariant functions (see [11]). This classical notion of a Gel'fand pair can be formulated as the cocommutativity of the coproduct

$$\Delta(f)(g,h) := \int_K f(gkh)d\mu_K(k) \quad (\mu_K - \ Haar\ measure\ for\ K) \tag{1}$$

on the space of all K-bi-invariant functions on G. Considering such functions as functions on the set of double cosets $Q = K\backslash G/K$, one can rewrite (1) in the following form:

$$\Delta(f)(p,r) = \int_Q K(p,r,s)f(s)d\mu_Q(s) \qquad (p,r \in Q), \tag{2}$$

where $K(.,.,.)$ is some positive kernel, μ_Q is some positive Borel measure on Q (which can depend on p and r in general case). A function $\chi_\alpha(.)$ on Q (α is classifying parameter) is called a character of the coalgebra given by (2) if it satisfies a product formula:

$$\int_Q K(p,r,s)\chi_\alpha(s)d\mu_Q(s) = \chi_\alpha(p)\chi_\alpha(r) \qquad (p,r \in Q). \tag{3}$$

We will say that the coproduct (2) defines a hypergroup structure on the algebra of K-bi-invariant functions on G with the pointwise multiplication. One can find a discussion of hypergroups in [5],[6],[13],[22],[26] and in references given there. In many cases the χ_α are well known special functions. Very often we have a similar formula with respect to α - dual product formula. It shows that χ_α is also a character of a dual hypergroup by the variable α.

In this paper we consider double cosets of compact quantum group with respect to its subgroup and distinguish cases of a Gel'fand pair and a strict Gel'fand pair. We show that every strict Gel'fand pair of compact quantum groups generates a normal commutative hypercomplex system with compact basis [5],[6] and a commutative discrete hypergroup [13], which are in duality to one another, consider corresponding examples and describe characters of hypergroups in terms of q-orthogonal polynomials.

After this paper had gone to press, the essential development of the subject took place. On the one hand, Gel'fand pairs for non compact quantum groups were considered (see, for example, [29]). On the other hand, one can consider a notion of a quantum subgroup of a quantum group from more general point of view then in this paper, using a notion of a coideal (see, for example, [9],[12],[14],[15],[20], [21],[23],[28]). This permits to apply the Gel'fand pair approach to exceptionally interesting classes of q-special functions such as Macdonalds and Askey-Wilson polynomials and Jacksons q-Bessel functions. This development is described in the survey [28].

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2 Double cosets of quantum groups

2.1. Let $\mathbf{H}:=(H,d,1,\Delta,\varepsilon,S), \widehat{\mathbf{H}} := (\widehat{H},\hat{d},\hat{1},\hat{\Delta},\hat{\varepsilon},\hat{S})$ be two Hopf algebras over C [1], with multiplications $d,\hat{d}$, units $1,\hat{1}$, comultiplications $\Delta,\hat{\Delta}$, counits $\varepsilon,\hat{\varepsilon}$, antipodes $S,\hat{S}$

Definition 1 *We say that $\mathbf{H},\widehat{\mathbf{H}}$ are in duality, if there exists a doubly non-degenerate pairing $\langle\cdot,\cdot\rangle$: $H\times\widehat{H}\to C$ such that:*

$$\langle 1,\zeta\rangle=\hat{\varepsilon}(\zeta),\ \langle ab,\zeta\rangle=\langle a\otimes b,\hat{\Delta}(\zeta)\rangle,\ \langle\Delta(a),\zeta\otimes\eta\rangle=\langle a,\zeta\eta\rangle,$$

$$\langle a,\hat{1}\rangle=\varepsilon(a),\ \langle S(a),\zeta\rangle=\langle a,\hat{S}(\zeta)\rangle \qquad (\forall a,b\in H,\ \zeta,\eta\in\widehat{H}).$$

We can define elements $\zeta * a := (id\otimes\zeta)\circ\Delta(a)$, $\quad a*\zeta := (\zeta\otimes id)\circ\Delta(a)$, where the pairing is used in the first, respectively second part of the tensor product. It is possible to rewrite the last equalities as $\langle\zeta * a,\eta\rangle=\langle a,\eta\zeta\rangle$, $\langle a*\zeta,\eta\rangle=\langle a,\zeta\eta\rangle$. These operations yield left and right algebra actions of $\widehat{H}$ on H:

$$(\zeta\eta)*a=\zeta*(\eta*a),\ a*(\zeta\eta)=(a*\zeta)*\eta \qquad (\forall a\in H,\ \eta,\zeta\in\widehat{H}).$$

Now let $\mathbf{H},\widehat{\mathbf{H}}$ be two Hopf algebras in duality, $\zeta\in\widehat{H}$. In what follows we will suppose that another pair $(\mathbf{H}_1,\widehat{\mathbf{H}}_1)$ of Hopf algebras in duality exists together with an epimorphism $\pi:\mathbf{H}\to\mathbf{H}_1$ and embedding $i:\widehat{\mathbf{H}}_1\to\widehat{\mathbf{H}}$ such that $\langle\pi(a),\zeta\rangle=\langle a,i(\zeta)\rangle$ $\quad(\forall a\in H,\zeta\in\widehat{H}_1)$. Left and right coactions $\Delta^l:=(\pi\otimes id)\circ\Delta$, $\Delta^r:=(id\otimes\pi)\circ\Delta$ of $\mathbf{H}_1$ on H define the subsets of left-, right- and bi-ivariant elements: $H_1\backslash H:=\{h\in H|\Delta^l(h)=1_1\otimes h\}$, $H/H_1:=\{h\in H|\Delta^r(h)=h\otimes 1_1\}$, $H_1\backslash H/H_1:=H_1\backslash H\cap H/H_1$. All these sets are evidently unital algebras. Let an invariant integral ν_1 (such that $\nu_1(1)=1$) on Hopf algebra $\mathbf{H}_1$ [1] exist (it always exists when $\mathbf{H}_1$ is a compact quantum group in the sense of [34]). Then we can introduce two projections $\pi^l:=(\nu_1\circ\pi\otimes id)\circ\Delta$, $\pi^r:=(id\otimes\nu_1\circ\pi)\circ\Delta$ from H to $H_1\backslash H$ and H/H_1 correspondingly. They commute and $\pi^r\circ\pi^l$ is a projection from H to $H_1\backslash H/H_1$ (see [7],[19]). A new coproduct may be introduced on $H_1\backslash H/H_1$:

$$\tilde{\Delta}:=(id\otimes\nu_1\circ\pi\otimes id)\circ(id\otimes\Delta)\circ\Delta \tag{4}$$

This definition is a generalization of (1) for Hopf algebra case.

Theorem 1 *Let a mapping* $\tilde{\Delta}$ *be defined by (4). Then:*

(a) $\tilde{\Delta}$ *maps* $H_1\backslash H/H_1$ *into* $H_1\backslash H/H_1 \otimes H_1\backslash H/H_1$*;*

(b) $\tilde{\Delta}$ *is coassociative, i.e.* $(id \otimes \tilde{\Delta}) \circ \tilde{\Delta} = (\tilde{\Delta} \otimes id) \circ \tilde{\Delta}$*;*

(c) ε *is a counit with respect to* $\tilde{\Delta}$*:* $(\varepsilon_1 \otimes id) \circ \tilde{\Delta} = (id \otimes \varepsilon_1) \circ \tilde{\Delta} = id$*;*

(d) *if* ν *is an invariant integral on* $\mathbf{H}$*, then* ν *is invariant with respect to* $\tilde{\Delta}$*:*

$$(\nu \otimes id) \circ \tilde{\Delta}(h) = (id \otimes \nu) \circ \tilde{\Delta}(h) = \nu(h) \cdot 1;$$

(e) *the following relation holds:* $\tilde{\Delta} \circ S = \Pi \circ (S \otimes S) \circ \tilde{\Delta}$.

PROOF. a) Evidently, $\tilde{\Delta} = (id \otimes \pi^l)\Delta = (\pi^r \otimes id)\Delta$. On the other hand, $(id \otimes \pi^r)\Delta = \Delta \circ \pi^r, (\pi^l \otimes \pi^l)\Delta = \Delta \circ \pi^l$. So for every $h \in H_1\backslash H/H_1$ we have $\tilde{\Delta}(h) \in H_1\backslash H/H_1 \otimes H$. Similarly we see that $\tilde{\Delta}(h) \in H \otimes H_1\backslash H/H_1$. b) Both sides of needed equality coincide with $(\pi^r \otimes id \otimes \pi^l)(\Delta \otimes id)\Delta$. c) $(\varepsilon \otimes id)\tilde{\Delta} = (\varepsilon \otimes \pi^l)\Delta = \pi^l$, so that ε is right counit. Similarly one can see that it is also left counit. d) Replacing ε by ν, we can prove this statement exactly as previous. e) This is implied by the following chain of equalities: $\tilde{\Delta} \circ S = (id \otimes \pi^l)\Delta \circ S = (id \otimes \pi^l)\Pi(S \otimes S)\Delta = (id \otimes \nu_1 \circ \pi \otimes id)$ $(\Pi \otimes id)(id \otimes \Pi)(S \otimes S \otimes S)(\Delta \otimes id)\Delta = \Pi(S \otimes S)(id \otimes \nu_1 \circ S_1 \circ \pi \otimes id)(\Delta \otimes id)\Delta = \Pi(S \otimes S)\tilde{\Delta}$. □

Two Hopf *-algebras $\mathbf{H}, \widehat{\mathbf{H}}$ are said to be in duality, if they are in duality as Hopf algebras and

$$\zeta^*(f) = \overline{\zeta(S(f^*))} \quad \forall \zeta \in \widehat{\mathbf{H}},\ f \in \mathbf{H},$$

where the same symbol denotes the involution in $\mathbf{H}$ and in $\widehat{\mathbf{H}}$. In what follows, we will be considering $\mathbf{H}, \mathbf{H}_1$ as Hopf *-algebras (see, for example, [25],[32],[34]) with the Hopf algebra structure and the involution *, π as an epimorphism of Hopf *-algebras, ν_1 as a state on the *-algebra H_1. Then $H_1\backslash H$, H/H_1 and $H_1\backslash H/H_1$ will be unital *-algebras, $\pi^l, \pi^r, \tilde{\Delta}$ map the cone of positive elements into the cones of positive elements of the corresponding *-algebras.

Definition 2 *A pair of Hopf algebras (resp. *-Hopf algebras)* $(\mathbf{H}, \mathbf{H}_1)$ *is called a Gel'fand pair if the coproduct* $\tilde{\Delta}$ *is cocommutative. A Gel'fand pair is called strict if the algebra* $H_1\backslash H/H_1$ *is commutative.*

2.2. Now let $\mathbf{H}$ be *-Hopf algebra associated with a compact quantum group and $\widehat{\mathbf{H}}$ is its algebraic dual. We know [34] that H can be represented as

$$H = \sum_{\alpha} \sum_{i,j=1}^{d_\alpha} \mathbf{C} u^{\alpha}_{i,j}, \tag{5}$$

where $u^{\alpha}_{i,j}$ are matrix elements of d_α- dimensional unitary corepresentation of $\mathbf{H}$ ($d_\alpha < \infty$ for all α running in some discrete set $\widehat{Q}$) and there exists an invariant integral ν on H, which is a state and such that α-sum in (5) defines an orthogonal decomposition in the sense of the inner product given by $\langle f, g \rangle := \nu(f \cdot g^*)$ after a suitable choice of an orthonormal basis for each representation space. In this case, the comodules $H_1\backslash H$, H/H_1 and also $H_1\backslash H/H_1$ may be given by

$$H_1\backslash H = \sum_{\alpha}\sum_{i=1}^{d'_\alpha}\sum_{j=1}^{d_\alpha} \mathbf{C}u^\alpha_{i,j},\ H/H_1 = \sum_{\alpha}\sum_{i=1}^{d_\alpha}\sum_{j=1}^{d'_\alpha} \mathbf{C}u^\alpha_{i,j},$$

$$H_1\backslash H/H_1 = \sum_{\alpha}\sum_{i,j=1}^{d'_\alpha} \mathbf{C}u^\alpha_{i,j}$$

where $d'_\alpha \le d_\alpha$ for all α. A notion of a Gel'fand pair for compact quantum groups was introduced in [19] as a pair $(\mathbf{H}, \mathbf{H}_1)$ with an epimorphism $\pi : \mathbf{H} \to \mathbf{H}_1$, such that for any irreducible unitary matrix corepresentation of $\mathbf{H}$, the dimension of the space of bi-invariant matrix elements is not greater then 1.

Lemma 1 *A pair of compact quantum groups $(\mathbf{H}, \mathbf{H}_1)$ with an epimorphism $\pi : \mathbf{H} \to \mathbf{H}_1$ is a Gel'fand pair in the sense of Definition 2, iff for any irreducible unitary matrix corepresentation of $\mathbf{H}$, the dimension of the space of bi-invariant matrix elements is not greater then 1.*

PROOF. Suppose that $2 \le d_\beta$ for some fixed $\beta \in \hat{Q}$. Set $\eta_1(u^\beta_{1,2}) := 1,\ \eta_1(u^\alpha_{i,j}) := 0$ otherwise and $\eta_2(u^\beta_{2,1}) := 1,\ \eta_2(u^\alpha_{i,j}) := 0$ otherwise. One can check that $\eta_1, \eta_2 \in (H_1\backslash H/H_1)^* \cap \widehat{H}_1$. Direct calculations show that $\langle\tilde{\Delta}(u^\beta_{1,1}), \eta_1 \otimes \eta_2\rangle \ne \langle\tilde{\Delta}(u^\beta_{1,1}), \eta_2 \otimes \eta_1\rangle$, i.e., $\tilde{\Delta}$ is not cocommutative. Conversely, if $d_\alpha = 1\ \forall\alpha \in \hat{Q}$, then $\tilde{\Delta}$ is obviously cocommutative. □

3 Connections with hypercomplex systems and hypergroups

3.1. We will use notions of a spatial tensor product for C^*-algebras, a unital Hopf C^*-algebra, a morphism, and a counit for unital Hopf C^*-algebras, as well as notions of a coaction of a unital Hopf C^*-algebra on a unital C^*- algebra and finite Haar measure on a unital Hopf C^*-algebra (see[4],[10],[34]). If $\mathbf{H}$ is a unital Hopf C^*-algebra, then the coproduct defines a structure of a Banach algebra in the conjugate space H^* for the C^*-algebra H:

$$\tilde{\omega} * \omega := (\tilde{\omega} \otimes \omega) \circ \Delta, \quad \forall \omega, \tilde{\omega} \in H^*.$$

$\mathbf{H}$ has a counit if and only if H^* is a unital algebra.

As in Section 2, we denote $\forall a \in H, \omega \in H^*$:

$$\omega * a := (id \otimes \omega) \circ \Delta(a),\ a * \omega := (\omega \otimes id) \circ \Delta(a).$$

Let ν be finite Haar measure on a unital Hopf C^*-algebra $\mathbf{H}$. One can introduce by means of GNS-construction a structure of the Hilbert space $L_2(H,\nu)$ and the corresponding representation of Λ_ν of H in this space. For every compact quantum group the completion of the initial Hopf *-algebra with respect to the C^{**}-norm $|\cdot| = \sup_\rho ||\rho(\cdot)||$,