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Introduction

Woronowicz ([W1], [W2]) defined a family (in the set-theoretical sense) of quantum groups $(SU_\mu(2))_{\mu\in\mathbf{R}^*}$. (For $\mu = 1$, the C^*-algebra A_μ underlying $SU_\mu(2)$ is merely the algebra $C(SU(2))$ of continuous functions from the classical group $SU(2)$ into $\mathbf{C}$).

"Forgetting" the group structure of $SU(2)$, Sheu ([S1]) used the Weyl calculus to construct a continuous deformation of the Poisson structure of $C^\infty(SU(2))$, where the fibres are precisely the C^*-algebras A_μ.

Unifying these two points of view, we shall do the following :

(§1) put Woronowicz's $SU_\mu(2)$'s together into a continuous field of quantum groups,

(§2) construct a deformation of Poisson-$SU(2)$ in the underlying continuous field of C^*-algebras A_μ,

(§3) prove that such a deformation is unique among deformations fulfilling suitable requirements,

(§4) prove that Sheu's deformation fulfills these requirements, and compare it in detail with our deformation.

Paragraphs 2, 3 and 4 will be achieved by working, as Sheu did, at the more elementary level of Poisson-deformations of the disc, which is a "slice" of $SU(2)$.

1 Continuous structure on the family of quantum groups $SU_\mu(2)$

Definition 1.1 *([W1], [W2]) For any $\mu \in \mathbf{R}$, A_μ is the enveloping C^*-algebra of the involutive $\mathbf{C}$-algebra $\mathcal{A}_\mu$ defined by two generators α_μ, γ_μ and relations :*

$$\begin{aligned} \alpha_\mu^*\alpha_\mu + \gamma_\mu^*\gamma_\mu &= 1 \quad (1_\mu) \\ \alpha_\mu\alpha_\mu^* + \mu^2\gamma_\mu^*\gamma_\mu &= 1 \quad (2_\mu) \end{aligned} \qquad \begin{aligned} \gamma_\mu^*\gamma_\mu &= \gamma_\mu\gamma_\mu^* \quad (3_\mu) \\ \alpha_\mu\gamma_\mu &= \mu\gamma_\mu\alpha_\mu \quad (4_\mu) \\ \alpha_\mu\gamma_\mu^* &= \mu\gamma_\mu^*\alpha_\mu \quad (5_\mu) \end{aligned}$$

and if $\mu \neq 0$, the quantum group $SU_\mu(2)$ is defined by the unitary matrix

$$\begin{pmatrix} \alpha_\mu & -\mu\gamma_\mu^* \\ \gamma_\mu & \alpha_\mu^* \end{pmatrix}$$

In order to endow the family of these $SU_\mu(2)$'s with a structure of continuous field of quantum groups ([B1], [B2]), we just have to endow the family $(A_\mu)_{\mu\in\mathbf{R}^*}$ with a structure of continuous field of C^*-algebras in such a way that the sections $\mu \mapsto \alpha_\mu$, $\mu \mapsto \gamma_\mu$ are continuous. (We shall even do a little more : this field will also be defined at $\mu = 0$).

Definition 1.2 *A is the universal C^*-algebra defined by three generators α, γ, μ and relations :*

$$\begin{aligned} \alpha^*\alpha + \gamma^*\gamma &= 1 \quad (1) \\ \alpha\alpha^* + \mu^2\gamma^*\gamma &= 1 \quad (2) \end{aligned} \qquad \begin{aligned} \gamma^*\gamma &= \gamma\gamma^* \quad (3) \\ \alpha\gamma &= \mu\gamma\alpha \quad (4) \\ \alpha\gamma^* &= \mu\gamma^*\alpha \quad (5) \end{aligned}$$

$$\begin{aligned} &\mu \quad \textit{commutes with} \quad \alpha, \gamma \quad (6) \\ &-1 \leq \mu \leq 1 \quad (7). \end{aligned}$$

$\mathcal{A}$ is the *-subalgebra generated by α, γ, μ.

The restriction of the parameter μ to $[-1;1]$ is harmless since for $\mu \neq 0$, there is an isomorphism of *quantum groups* (not only of C^*-algebras) between $SU_\mu(2)$ and $SU_{1/\mu}(2)$ (sending α_μ to $\alpha_{1/\mu}{}^*$ and γ_μ to $-\frac{1}{\mu}\gamma_{1/\mu}{}^*$) : using this isomorphism it is then easy to extend to $\mathbf{R}$ the field on $[-1;1]$ which we are going to construct.

Moreover, such a restriction of the parameter is necessary, otherwise the generator μ would not be bounded, hence the involutive algebra defined by these generators and relations would not have a C^*-envelope.

We shall construct a field of C^*-algebras over $[-1;1]$, using the natural morphism from $C([-1;1])$ into the center of A. (This morphism is given by relations (6) and (7)). By a slight generalization of the Dauns-Hofmann theorem, proved by Dupré and Gilette ([DG], proposition 1.3 and corollary 2.2) and quoted in [Ri], there is a unique upper semi-continuous field related to the $C([-1;1])$-C^*- algebra A in the following way.

Definition 1.3 *ξ is the upper semi-continuous field of C^*-algebras on $[-1;1]$ such that :*

- *the fiber of ξ at x is A/xA (x denotes here both a point in $[-1;1]$ and the ideal of functions in $C([-1;1])$ vanishing at this point)*
- *the total space $\sqcup_{x\in[-1;1]}A/xA$ of ξ is endowed with a topology such that the continuous sections of ξ are the sections of the form $x \mapsto a \bmod xA$, for any $a \in A$.*

Using the universal properties of A and of the A_μ's, one easily proves the following relationship between our field ξ and Woronowicz's family $(A_\mu)_{\mu\in[-1;1]}$.

Proposition 1.4 *For any $\mu \in [-1;1]$, the fiber at μ of the field ξ is naturally isomorphic to the C^*-algebra A_μ. This family of isomorphisms identifies the two continuous sections of the field ξ associated to $\alpha, \gamma \in A$ with the two sections $\mu \mapsto \alpha_\mu, \mu \mapsto \gamma_\mu$ of the family $(A_\mu)_{\mu\in[-1;1]}$.*

Before introducing another field ζ with more elementary fibers, and proving the (lower) continuity of both fields ξ and ζ, let us first get rid of the case $\mu \leq 0$: we shall prove that the study of $\xi_{|[-1,0]}$ may be reduced to the study of $\xi_{|[0;1]}$ (and conversely), by a property with "fractal" flavour.

Proposition 1.5 *Let $\varepsilon_{i,j}$ $(1 \leq i,j \leq 2)$ be the canonical generators of $M_2(\mathbf{C})$ and $i_\mu : A_{-\mu} \to M_2(\mathbf{C}) \otimes A_\mu$ the morphism defined by :*

$$i_\mu(\alpha_{-\mu}) = (\varepsilon_{1,2} + \varepsilon_{2,1}) \otimes \alpha_\mu, \qquad i_\mu(\gamma_{-\mu}) = (\varepsilon_{1,1} - \varepsilon_{2,2}) \otimes \gamma_\mu.$$

For any $\mu \in \mathbf{R}$, i_μ is an embedding.

Proof. Let D be the subalgebra of $M_2(\mathbf{C}) \otimes M_2(\mathbf{C})$ generated by the two elements $P = \varepsilon_{1,1} \otimes \varepsilon_{1,1} + \varepsilon_{2,2} \otimes \varepsilon_{2,2}$ and $Q = \varepsilon_{1,2} \otimes \varepsilon_{1,2} + \varepsilon_{2,1} \otimes \varepsilon_{2,1}$ and similarly, D' the subalgebra generated by $P' = \varepsilon_{1,1} \otimes \varepsilon_{2,2} + \varepsilon_{2,2} \otimes \varepsilon_{1,1}$ and $Q' = \varepsilon_{1,2} \otimes \varepsilon_{2,1} + \varepsilon_{2,1} \otimes \varepsilon_{1,2}$. Let $\varphi : D \to \mathbf{C}$ be the morphism such that $\varphi(P) = \varphi(Q) = 1$. One easily checks that the image of $j = (\mathrm{id}_{M_2(\mathbf{C})} \otimes i_{-\mu}) \circ i_\mu$ is included in $(D \oplus D') \otimes A_{-\mu}$ and that $((\varphi \oplus 0) \otimes \mathrm{id}_{A_{-\mu}}) \circ j = \mathrm{id}_{A_{-\mu}}$.

We shall now reduce the study of the A_μ's (quantum $SU(2)$) to the study of more elementary C^*-algebras B_μ (quantum disc). Let us recall the two results which naturally led us to this reduction.

Theorem 1.6 *([W2] appendix 2) A_μ is isomorphic to A_0, for any $\mu \in]-1, 1[$.*

The isomorphism $T_\mu : A_\mu \xrightarrow{\sim} A_0$ was defined by Woronowicz as follows :

$$\begin{aligned} T_\mu(\alpha_\mu) &= \textstyle\sum_{n=0}^{\infty} \frac{(1-\mu^2)\mu^{2n}}{\sqrt{1-\mu^{2n+2}}+\sqrt{1-\mu^{2n}}} \alpha_0^{*n} \alpha_0^{n+1} \\ \text{and} \quad T_\mu(\gamma_\mu) &= \textstyle\sum_{n=0}^{\infty} \mu^n \alpha_0^{*n} \gamma_0 \alpha_0^n \end{aligned}$$

Theorem 1.7 *([S1] proposition 1.1) Let*

$$0 \longrightarrow C(D) \longrightarrow C(\overline{D}) \xrightarrow{\sigma_1} C(\mathbf{T}) \longrightarrow 0$$

be the exact sequence of the unit disc and

$$0 \longrightarrow \mathcal{K} \longrightarrow C^*(\mathcal{S}) \xrightarrow{\sigma_0} C(\mathbf{T}) \longrightarrow 0$$

be the Tœplitz exact sequence. Set $B_1 = C(\overline{D})$ and $B_0 = C^*(\mathcal{S})$. For $\mu = 1$ or 0, A_μ is isomorphic to the algebra of continuous functions $f : \mathbf{T} \to B_\mu$ such that $\sigma_\mu(f(u))$ does not depend on $u \in \mathbf{T}$.

For $\mu = 1$, the isomorphism consists in identifying $SU(2)$ with a family of discs $(D_u)_{u \in \mathbf{T}}$, glued together along their boundary circle :

$$(u, Z) \in \mathbf{T} \times \overline{D} \text{ is identified to } \begin{pmatrix} Z & -\overline{u}c \\ uc & \overline{Z} \end{pmatrix}, \text{ with } c = \sqrt{1 - |Z|^2}.$$

(This "slicing" of $SU(2)$ is compatible with the Poisson structure, cf §3 and 4).

For $\mu = 0$, let us recall the Tœplitz exact sequence. $C^*(\mathcal{S})$ is the C^*-algebra generated by the unilateral shift operator $\mathcal{S}$. $\mathcal{S}\mathcal{S}^*$ is equal to $1 - p$, p being a rank one projection. The closed ideal of $C^*(\mathcal{S})$ generated by p is the algebra $\mathcal{K}$ of compact operators, and $C^*(\mathcal{S})/\mathcal{K}$ is isomorphic to $C(\mathbf{T})$, the isomorphism sending the unitary generator $(\mathcal{S} \bmod \mathcal{K}) \in C^*(\mathcal{S})/\mathcal{K}$ to $\mathrm{id}_{\mathbf{T}}$.

In both cases $\mu = 1$ or 0, the embedding $A_\mu \to C(\mathbf{T}, B_\mu)$ sends

α_μ to $(u \mapsto \overline{\alpha}_\mu)$ and γ_μ to $(u \mapsto u\overline{\gamma}_\mu)$, with

$$\begin{aligned} \overline{\alpha}_1 &= (Z \mapsto Z), & \overline{\gamma}_1 &= (Z \mapsto \sqrt{1 - |Z|^2}) \\ \overline{\alpha}_0 &= \mathcal{S}^*, & \overline{\gamma}_0 &= p. \end{aligned}$$

Using the proof of theorem 1.6, one gets the following "generalization" of theorem 1.7 for free (we pass from the case $\mu = 0$ to the "more general" case $|\mu| < 1$ by a rather silly renaming ; the only nontrivial assertion of the following corollary is the first one, which justifies this renaming).

Corollary 1.8 *For $|\mu| < 1$, let us denote by*

- $\overline{\alpha}_\mu, \overline{\gamma}_\mu$ *the elements of $C^*(\mathcal{S})$ defined as series in $\overline{\alpha}_0, \overline{\gamma}_0$ by the same formulas as in theorem 1.6, where $T_\mu(\alpha_\mu), T_\mu(\gamma_\mu)$ were defined as series in α_0, γ_0*
- $\mathcal{B}_\mu$ *the involutive subalgebra generated by $\overline{\alpha}_\mu, \overline{\gamma}_\mu$ and*
- B_μ *its closure in $C^*(\mathcal{S})$.*

For $|\mu| < 1$, B_μ is equal to $C^*(\mathcal{S})$ and the closed ideal of B_μ generated by $\overline{\gamma}_\mu$ is $\mathcal{K}$. Moreover, for $-1 < \mu \leq 1$, there is a morphism $\sigma_\mu : B_\mu \to C(\mathbf{T})$ such that $\sigma_\mu(\overline{\alpha}_\mu^*) = \mathrm{id}_\mathbf{T}$ and such that the sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow B_\mu \xrightarrow{\sigma_\mu} C(\mathbf{T}) \longrightarrow 0$$

is exact, and A_μ is isomorphic to the algebra of continuous functions $f : \mathbf{T} \to B_\mu$ such that $\sigma_\mu(f(u))$ does not depend on $u \in \mathbf{T}$.

Remark. For $\mu = 1$, the morphism $\sigma_1 : C(\overline{D}) \to C(\mathbf{T})$ which we are choosing is *not the mere restriction* but (in order to make the notations fit together) $\sigma_1(f)(v) = f(\overline{v})$. This remark will be important in lemma 4.5.

Since $\overline{\gamma}_0 = p \geq 0$, the definition of $\overline{\gamma}_\mu$ as a series makes it self adjoint. If $\mu \geq 0$ we even get : $\overline{\gamma}_\mu \geq 0$, hence (under the identification of A_μ given above) $|\gamma_\mu| = \left|(u \mapsto u\overline{\gamma}_\mu)\right| = (u \mapsto \overline{\gamma}_\mu)$. Using this fact and the universal property of A_μ, one easily proves the following proposition.

Proposition 1.9 *For $0 \leq \mu \leq 1$, B_μ is isomorphic to the universal C^*-algebra defined by generators $\overline{\alpha}_\mu, \overline{\gamma}_\mu$ and relations :*

the relations (1_μ)–(5_μ) (cf definition 1.1)

the additional relation : $\overline{\gamma}_\mu \geq 0$.

Remark. Instead of adding a relation and looking at B_μ as a quotient of A_μ, one may also prove (but this will not be used) that B_μ is isomorphic to the C^*-subalgebra of A_μ generated by α_μ, and characterize B_μ as the universal C^*-algebra defined by one generator α_μ and one relation (deduced from relations (1_μ) and (2_μ) by eliminating γ_μ) ([NN]).

Paraphrasing definitions 1.2 and 1.3 and proposition 1.4, we can now define the field of B_μ's.