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1 Introduction, and some history.

Let G be a locally compact group, and H be a closed subgroup. Viewing $L^1(G)$ as a two-sided ideal in the measure algebra $M(G)$, and viewing elements of $L^1(H)$ as measures on G supported inside H, we obtain an action of $L^1(H)$ on $L^1(G)$ as double centralizers. It is easy to check (see e.g. proposition 4.1 in [Rie]) that this action extends to an action of the full group C^*-algebra $C^*(H)$ as double centralizers on C^*(G); this corresponds to a $*$-homomorphism $j_H : C^*(H) \to M(C^*(G))$, where $M(C^*(G))$ denotes the multiplier C^*-algebra of $C^*(G)$. We now quote from p. 209 of Rieffel's Advances paper [Rie]:

It does not seem to be known whether this homomorphism j_H is injective. It will be injective if and only if every unitary representation of H is weakly contained in the restriction to H of some unitary representation of G [Fe2]. J.M.G. Fell has pointed out to us that the example that he gave in which this appeared to fail (p. 445 of [Fe2]) depended on the completeness of the classification of the irreducible representations of $SL_3(\mathbb{C})$ given in [GeN], and there is now some doubt that this classification is complete [Ste].

Probably this quotation requires some word of explanation. In [Fe2], Fell studies extensions to the topological framework of Frobenius reciprocity for finite groups. Thus he introduces a list of weak Frobenius properties, the last and weakest one being (WF3):

The locally compact group G satisfies property (WF3) if, for any closed subgroup H of G, every representation σ in the dual $\hat{H}$ is weakly contained in the restriction $\pi|_H$ of some unitary representation π of G.

Property (WF3) is indeed equivalent to the injectivity of j_H for any closed subgroup H; for completeness, we shall give a proof in Proposition 2.1 below. In §6 of [Fe2], Fell wishes to show that even (WF3) may fail, by taking $G = SL_3(\mathbb{C})$ and $H = SL_2(\mathbb{C})$; to this end he appeals to the incomplete description of $\hat{G}$ given in [GeN]; Fell's proof was recently corrected in Remark 1.13(i) of [BLS].

In this paper, we take for G a semi-simple Lie group with finite centre and without compact factor, and as closed subgroup a lattice Γ. In section 3, we prove:

THEOREM 1.1 *Let G be a semi-simple Lie group without compact factors, with finite centre and with Kazhdan's property (T). Let Γ be an irreducible lattice in G, and let σ be a non-trivial irreducible unitary representation of Γ of finite dimension n. Then σ*

determines a direct summand of $C^*(\Gamma)$ which is contained in the kernel of $j_\Gamma : C^*(\Gamma) \to M(C^*(G))$; this direct summand is isomorphic to the algebra $M_n(\mathbb{C})$ of n-by-n matrices.

If G is a non-compact *simple* Lie group with finite centre, then G has property (T) unless G is locally isomorphic either to $SO_o(n,1)$ or $SU(n,1)$ (see [HaV]). For these two families, we prove in section 4:

THEOREM 1.2 *Let G be locally isomorphic either to $SO_o(n,1)$ or $SU(n,1)$, for some $n \geq 2$. Let Γ be a lattice in G. Denote by $\hat{\Gamma}_f$ the set of (classes of) irreducible, finite-dimensional unitary representations of Γ. If the trivial representation 1_Γ is not isolated in $\hat{\Gamma}_f$ (for the induced Fell-Jacobson topology), then infinitely many elements of $\hat{\Gamma}_f$ are not weakly contained in the restriction to Γ of any unitary representation of G. In particular $j_\Gamma : C^*(\Gamma) \to M(C^*(G))$ is not injective.*

In view of Theorems 1.1 and 1.2, it seems natural to formulate the following

Conjecture. If Γ is a lattice in a non-compact semi-simple Lie group G, then $j_\Gamma : C^*(\Gamma) \to M(C^*(G))$ is not injective.

This conjecture means that, if ρ is a representation of G which is faithful on $M(C^*(G))$ (e.g. take for ρ either the universal representation of G, or the direct sum of all its irreducible representations), then $\rho|_\Gamma$ is never faithful on $C^*(\Gamma)$; this has bearing on a question of de la Harpe in his paper in these Proceedings (see immediately after Problem 13 in [Har]). In §5, we give examples of lattices Γ in $SO_o(n,1)$ or $SU(n,1)$ such that 1_Γ is not isolated in $\hat{\Gamma}_f$; this is the case for any lattice in $SL_2(\mathbb{R})$, any non-uniform lattice in $SL_2(\mathbb{C})$, and any arithmetic lattice in $SO_o(n,1)$ for $n \neq 3$, 7.

In the final §6, we come back to property (WF3) and show that it always fails for almost connected, non-amenable groups:

THEOREM 1.3 *Let G be an almost connected, locally compact group. The following properties are equivalent:*

(i) G has Fell's property (WF3);
(ii) G is amenable.

Observe that Theorem 1.3 cannot hold for any locally compact group. Indeed, any discrete group G satisfies property (WF3) since, given a subgroup H of G, one checks easily that $C^*(H)$ is a C^*-subalgebra of $C^*(G) = M(C^*(G))$.

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A word about terminology: as usual, semi-simple Lie groups are assumed to be connected and non-trivial; group representations are assumed to be unitary, strongly continuous, and on non-zero Hilbert spaces.

2 On multipliers of C^*-algebras.

For a C^*-algebra B, we denote by $M(B)$ its multiplier algebra.

PROPOSITION 2.1 *Let A, B be C^*-algebras, and let $j : A \to M(B)$ be a $*$-homomorphism. The following properties are equivalent:*

(i) j is one-to-one;

(ii) for any $\sigma \in \hat{A}$, there exists a non-degenerate $*$-representation π of B such that σ is weakly contained in $\tilde{\pi} \circ j$, where $\tilde{\pi}$ denotes the extension of π to $M(B)$;

(iii) any $\sigma \in \hat{A}$ is weakly contained in $\{\tilde{\pi} \circ j \mid \pi \in \hat{B}\}$.

Fell's property (WF3), mentioned in §1, is deduced from property (ii) above by taking $B = C^*(G)$ and $A = C^*(H)$, for any closed subgroup H of the locally compact group G.

Proof of Proposition 2.1. (i) $\Rightarrow$ (ii) Let us assume that j is injective, so that we may identify A with a C^*-subalgebra of $M(B)$. Let π be a faithful representation of B. It is known that the extension $\tilde{\pi}$ of π to $M(B)$ is also faithful ([Ped], 3.12.5). Thus any representation of A is weakly contained in the restriction of $\tilde{\pi}$ to A.

(ii) $\Rightarrow$ (iii) This follows from decomposition theory.

(iii) $\Rightarrow$ (i) Assume that (iii) holds. Fix a non-zero element x of A; choose $\sigma \in \hat{A}$ such that $\sigma(x) \neq 0$. Our assumption says that $Ker\,\sigma$ contains $\bigcap_{\pi\in\hat{B}} Ker\tilde{\pi} \circ j = Ker(\bigoplus_{\pi\in\hat{B}} \tilde{\pi} \circ j)$; in particular $x \notin Ker(\bigoplus_{\pi\in\hat{B}} \tilde{\pi} \circ j)$. It follows that $j(x) \neq 0$, i.e. that j is one-to-one.

3 Proof of theorem 1.1

We slightly generalize Theorem 1.1 in the following form:

THEOREM 3.1 *Let G be a non-compact semi-simple Lie group with finite centre and with Kazhdan's property (T). Let Γ be an irreducible lattice in G, and let σ be an irreducible representation of Γ of finite dimension n, which is not contained in the restriction to Γ of a unitary, finite-dimensional representation of G. Then σ determines a direct summand of $C^*(\Gamma)$ isomorphic to the algebra $M_n(\mathbb{C})$ of n-by-n matrices, which moreover is contained in the kernel of $j_\Gamma : C^*(\Gamma) \to M(C^*(G))$.*

Observe that Theorem 1.1 is an immediate consequence of Theorem 3.1: indeed, if G has no compact factor, then any unitary, finite-dimensional representation of G is trivial.

Proof of Theorem 3.1. Since G has property (T), so has Γ (see [HaV], Théorème 4 in Chapter 3). Let σ be an irreducible representation of Γ, of finite dimension n. By Theorem 2.1 in [Wan], σ is isolated in the dual $\hat{\Gamma}$, hence determines a direct sum decomposition of $C^*(\Gamma)$:

$$C^*(\Gamma) = J \oplus M_n(\mathbb{C})$$

where J is the C^*-kernel of σ.

We assume from now on that σ is not contained in the restriction to Γ of a unitary, finite-dimensional representation of G, and wish to prove that the direct summand $M_n(\mathbb{C})$ lies in the kernel of $j_\Gamma : C^*(\Gamma) \to M(C^*(G))$. Suppose by contradiction that j_Γ is non-zero on $M_n(\mathbb{C})$. Choose $\pi \in \hat{G}$ such that $\tilde{\pi} \circ j_\Gamma$ is non-zero, hence faithful on $M_n(\mathbb{C})$ (here $\tilde{\pi}$ denotes the extension of π to $M(C^*(G))$, as in Proposition 2.1). Then the C^*-kernel of $\tilde{\pi} \circ j_\Gamma$ is contained in J, which means that σ is weakly contained in the restriction $\pi|_\Gamma$. As σ is isolated in $\hat{\Gamma}$, this implies that σ is actually a subrepresentation of $\pi|_\Gamma$ (see Corollary 1.9 in [Wan]). Our assumption shows that π is infinite-dimensional. Two cases may occur:

(a) π is a discrete series representation of G (if any); this would imply that σ is an irreducible subrepresentation of the left regular representation of Γ, which in turn implies that Γ is finite - and this is absurd.

(b) π is not in the discrete series of G; then, by a result of Cowling and Steger (Proposition 2.4 in [CoS]), the restriction $\pi|_\Gamma$ is irreducible, which contradicts the fact that σ is a finite-dimensional subrepresentation.

With a contradiction reached in both cases, the proof of Theorem 3.1 is complete. We thank G. Skandalis for a helpful conversation that led to a more explicit version of Theorem 3.1.

Remark. Let us show that there are countably many finite-dimensional elements $\sigma \in \hat{\Gamma}$ satisfying the assumptions of Theorem 3.1.

Thus, let $G/Z(G)$ be the adjoint group of G; this is a linear group. Denote by Γ_1 the image of Γ in $G/Z(G)$; as a finitely generated linear group, Γ_1 is residually finite (see [Mal]); a non-trivial irreducible representation σ of Γ that factors through a finite quotient of Γ_1 cannot be contained in the restriction to Γ of a finite-dimensional unitary representation of G.

This argument shows that $Ker[j_\Gamma : C^*(\Gamma) \to M(C^*(G))]$ contains the C^*-direct sum of countably many matrix algebras.

4 The cases $SO_o(n, 1)$ and $SU(n, 1)$.

We begin with the following result, which is certainly known to many experts (see [Moo], Proposition 3.6; compare also with [Mar], Chap. III, (1.12), Remark 1).

PROPOSITION 4.1 *Let G be a simple Lie group with finite centre, and let Γ be a lattice in G. Denote by γ the quasi-regular representation of G on $L^2(G/\Gamma)$, and by γ_0 the restriction of γ to $L^2_o(G/\Gamma) = \{f \in L^2(G/\Gamma) | < f|1 >= 0\}$.*

(a) There exists $N \in \mathbb{N}$ such that the N-fold tensor product $\gamma_0^{\otimes N}$ is weakly contained in the left regular representation λ_G of G.

(b) The trivial representation 1_G is not weakly contained in γ_0.

Proof. (a) Suppose first that G has Kazhdan's property (T). Then, by Theorems 2.4.2 and 2.5.3 in [Cow], there exists $N \in \mathbb{N}$ such that $\pi^{\otimes N}$ is weakly contained in