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Ergodic Actions of Compact Matrix Pseudogroups on C*-algebras

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Dedicated to Professor Masamichi Takesaki on the ocasion of his 60th birthday

Let G be a compact group acting on a unital C^*-algebra $\mathcal{M}$. The action is said to be ergodic if the fixed point algebra $\mathcal{M}^G$ reduces to scalars. The first breakthrough in the study of such actions was the finiteness theorem of Høegh-Krohn, Landstad and Størmer [HLS]. They proved that the multiplicity of each $\pi \in \widehat{G}$ in $\mathcal{M}$ is at most $\dim(\pi)$ and the unique G-invariant state on $\mathcal{M}$ is necessarily a trace. When combined with Landstad's result [L] that finite-dimensionality for the spectral subspaces of actions of compact groups implies that the crossed product is a type I C^*-algebra (and in fact, as pointed out in [Wa1] is a direct sum of algebras of compact operators), the finiteness theorem shows that the crossed product of a unital C^*-algebra by an ergodic action of a compact group is necessarily equal to $\oplus_i \mathcal{K}(\mathcal{H}_i)$.

The study of such actions was essentialy pushed forward by Wassermann. He developed an outstanding machinery based on the notion of multiplicity maps, establishing a remarkable connection with the equivariant K-theory. This approach allowed him to prove, among other important things, the strong negative result that $SU(2)$ cannot act ergodically on the hyperfinite II_1 factor [Wa3].

The aim of this note is to study ergodic actions of Woronowicz's compact matrix pseudogroups on unital C^*-algebras, extending some of the previous results. In this insight we prove in §1 the analogue of the finiteness theorem. More precisely, if $G = (A, u)$ is a compact matrix pseudogroup acting ergodically on a unital C^*-algebra by a coaction $\sigma : \mathcal{M} \to \mathcal{M} \otimes A$ such that $\sigma(\mathcal{M})(1_{\mathcal{M}} \otimes A)$ is dense in $\mathcal{M} \otimes A$, then there is a decomposition of the *-algebra of σ-finite elements into isotypic subspaces $\mathcal{M}_0 = \oplus_{\alpha \in \widehat{G}} \mathcal{M}_\alpha$, orthogonal with respect to the scalar product induced on $\mathcal{M}$ by the unique σ-invariant state ω. Moreover, the spectral subspaces $\mathcal{M}_\alpha$ are finite dimensional and $\dim(\mathcal{M}_\alpha) \leq M_\alpha^2$, M_α being the *quantum dimension* of $\alpha \in \widehat{G}$. If the Haar measure is faithful on A, then $\mathcal{M}_0$ is dense in the GNS Hilbert space $\mathcal{H}_\omega$.

Although ω is not in general a trace, we prove the existence of a multiplicative linear map $\Theta : \mathcal{M}_0 \to \mathcal{M}_0$ such that $\omega(xy) = \omega(\Theta(y)x)$ for all $x \in \mathcal{M}, y \in \mathcal{M}_0$ and Θ is a scalar multiple of the modular operator F_α when restricted to each irreducible σ-invariant subspace of the spectral subspace $\mathcal{M}_\alpha$.

The crossed products by such coactions are studied in §2 where we prove, using the Takesaki-Takai type duality theorem of Baaj and Skandalis [BS], that they are isomorphic all the time to a direct sum of C^*-algebras of compact operators. As a corollary, if a compact matrix pseudogroup with underlying nuclear C^*-algebra acts ergodically on a unital C^*-algebra $\mathcal{M}$, then $\mathcal{M}$ follows nuclear.

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§1. The isotypic decomposition and finiteness of multiplicities for ergodic actions

We start with a couple of definitions.

Definition 1 ([BS]) *A coaction of a unital Hopf C^*-algebra (A, Δ_A) on a unital C^*-algebra $\mathcal{M}$ is a unital one-to-one *-homomorphism $\sigma : \mathcal{M} \to \mathcal{M} \otimes A$ (the tensor product will be all the time the minimal C^*-one) that makes the following diagram commutative*

$$\begin{tikzcd} \mathcal{M} \arrow[r, "\sigma"] & \mathcal{M} \otimes A \\ \mathcal{M} \otimes A \arrow[u, "\sigma"] \arrow[r, "\sigma \otimes id_A"] & \mathcal{M} \otimes A \otimes A \arrow[u, "id_{\mathcal{M}} \otimes \Delta_A"'] \end{tikzcd}$$

A C^*-algebra $\mathcal{M}$ with a coaction σ of (A, Δ_A) is called an A-algebra if σ is one-to-one and $\sigma(\mathcal{M})(1_{\mathcal{M}} \otimes A)$ is dense in $\mathcal{M} \otimes A$.

Definition 2 *The fixed points of the coaction $\sigma : \mathcal{M} \to \mathcal{M} \otimes A$ are the elements of $\mathcal{M}^\sigma = \{x \in \mathcal{M} \mid \sigma(x) = x \otimes 1_A\}$. The coaction σ is called ergodic if $\mathcal{M}^\sigma = \mathbf{C}1_{\mathcal{M}}$.*

We denote by $\mathcal{M}^*$ the set of continuous linear functionals on $\mathcal{M}$.

Definition 3 *$\phi \in \mathcal{M}^*$ is called σ-invariant if*
$(\phi \otimes \psi)(\sigma(x)) = (\phi \otimes \psi)(x \otimes 1_A) = \psi(1_A)\phi(x)$ *for all* $\psi \in A^*$.

Let $G = (A, u)$ be a compact matrix pseudogroup group with comultiplication $\Delta_A : A \to A \otimes A$, smooth structure $\mathcal{A}$ and coinverse $\kappa : \mathcal{A} \to \mathcal{A}$ (cf [Wor]). Then A^* is an algebra with respect to the convolution $\phi * \psi = (\phi \otimes \psi)\Delta_A$, $\phi, \psi \in A^*$ and there exists a unique state h on A, called the Haar measure of G, so that $\phi * h = h * \phi = \phi(1_A)h$ for all $\phi \in A^*$. Let $\mathcal{M}$ be a unital C^*-algebra which is an A-algebra via the coaction $\sigma : \mathcal{M} \to \mathcal{M} \otimes A$ and consider $\theta = (id_{\mathcal{M}} \otimes h)\sigma$.

Lemma 4 *i)* $\theta(x) \in \mathcal{M}^\sigma$ *for all* $x \in \mathcal{M}$. *Moreover* θ *is a conditional expectation from* $\mathcal{M}$ *onto* $\mathcal{M}^\sigma$.

ii) If σ *is ergodic, then* $\theta(x) = \omega(x)1_{\mathcal{M}}$ *and* ω *is the only* σ*-invariant state on* $\mathcal{M}$.

Proof. i) Note that

$$\begin{aligned}\sigma(\theta(x)) &= \sigma((id_{\mathcal{M}} \otimes h)(\sigma(x)) = (\sigma \otimes h)(\sigma(x))\\ &= (id_{\mathcal{M}} \otimes id_A \otimes h)(\sigma \otimes id_A)(\sigma(x))\\ &= (id_{\mathcal{M}} \otimes id_A \otimes h)(id_{\mathcal{M}} \otimes \Delta_A)(\sigma(x))\\ &= (id_{\mathcal{M}} \otimes (id_A \otimes h)\Delta_A)(\sigma(x)), \qquad x \in \mathcal{M}.\end{aligned}$$

Since $(id_A \otimes h)(\Delta_A(a)) = h * a = h(a)1_A$, $a \in A$ ([Wor, 4.2]), we obtain further :

$$\begin{aligned}&(\psi_1 \otimes \psi_2)((id_{\mathcal{M}} \otimes (id_A \otimes h)\Delta_A)(y \otimes a))\\ &\quad = (\psi_1 \otimes \psi_2)(y \otimes (id_A \otimes h)(\Delta_A(a)))\\ &\quad = (\psi_1 \otimes \psi_2)(y \otimes h(a)1_A) = \psi_1(y)\psi_2(1_A)h(a)\\ &\quad = (\psi_1 \otimes \psi_2)((id_{\mathcal{M}} \otimes h)(y \otimes a) \otimes 1_A),\end{aligned}$$

for all $y \in \mathcal{M}, a \in A, \psi_1 \in \mathcal{M}^*, \psi_2 \in A^*$. Therefore for $x \in \mathcal{M}, \psi_1 \in \mathcal{M}^*, \psi_2 \in A^*$ we have :

$$(\psi_1 \otimes \psi_2)(\sigma(\theta(x)) = (\psi_1 \otimes \psi_2)((id_{\mathcal{M}} \otimes h)\sigma(x) \otimes 1_A) = (\psi_1 \otimes \psi_2)(\theta(x) \otimes 1_A)$$

and consequently $\sigma(\theta(x)) = \theta(x) \otimes 1_A$ for all $x \in \mathcal{M}$. θ is a norm one projection since

$$\theta(x) = (id_{\mathcal{M}} \otimes h)(\sigma(x)) = (id_{\mathcal{M}} \otimes h)(x \otimes 1) = x, \qquad x \in \mathcal{M}^\sigma.$$

ii) Let $\psi \in A^*$. Then we get for all $x \in \mathcal{M}$:

$$\begin{aligned}(\omega \otimes \psi)(\sigma(x)) &= ((id_{\mathcal{M}} \otimes h)\sigma \otimes \psi)(\sigma(x))\\ &= (id_{\mathcal{M}} \otimes \psi)((id_{\mathcal{M}} \otimes h)\sigma \otimes id_A)(\sigma(x))\\ &= (id_{\mathcal{M}} \otimes \psi)(id_{\mathcal{M}} \otimes h \otimes id_A)((\sigma \otimes id_A)\sigma(x))\\ &= (id_{\mathcal{M}} \otimes \psi)(id_{\mathcal{M}} \otimes h \otimes id_A)((id_{\mathcal{M}} \otimes \Delta_A)\sigma(x))\\ &= (id_{\mathcal{M}} \otimes \psi)(id_{\mathcal{M}} \otimes (h \otimes id_A)\Delta_A)(\sigma(x))\\ &= \psi(1_A)(id_{\mathcal{M}} \otimes h)(\sigma(x)) = \psi(1_A)\omega(x),\end{aligned}$$

therefore ω is a σ-invariant state on $\mathcal{M}$. Finally, assume that ϕ is a σ-invariant state on $\mathcal{M}$. Then for all $x \in \mathcal{M}$:

$$\phi(x) = (\phi \otimes h)(\sigma(x)) = \phi((id_{\mathcal{M}} \otimes h)(\sigma(x))) = \phi(\theta(x)) = \phi(\omega(x)1_{\mathcal{M}}) = \omega(x). \qquad \square$$

Remarks. 1). The proof of the previous statement doesn't use the faithfulness of σ but only the equality $(\sigma \otimes id_A)\sigma = (id_{\mathcal{M}} \otimes \Delta_A)\sigma$.

2). Since the tensor product of two faithful completely positive maps is still faithful [T], it follows that if σ is one-to-one and h is faithful on A, then ω is a faithful state on $\mathcal{M}$.

3). Although one can easily pass from a compact matrix pseudogroup to the reduced one, which has faithful Haar measure, as indicated at page 656 in [Wor], it turns out that the Haar measure is faithful in several important examples (e.g. on commutative CMP, on reduced cocommutative CMP or, cf. [N], on $SU_\mu(N)$).

Denote by $\mathcal{H}_\omega$ the completion of $\mathcal{M}$ with respect to the inner product $\langle x,y\rangle_2 = \omega(y^*x)$ and let $\mathcal{M}$ acting on $\mathcal{H}_\omega$ in the GNS representation. Consider the C^*-Hilbert module $\mathcal{H}_\omega \otimes A$ with the A-valued inner product $\langle x_\omega \otimes a, y_\omega \otimes b\rangle_A = \omega(y^*x)b^*a$, for $x,y \in \mathcal{M}$, $a,b \in A$, which can be viewed as $\mathcal{M}\otimes A$ in the Stinespring representation of the completely positive map $\omega \otimes id_A$. Define also $V : \mathcal{H}_\omega \otimes A \to \mathcal{H}_\omega \otimes A$ by :

$$V(\sum_i (x_i)_\omega \otimes a_i) = \sum_i \sigma(x_i)(1_\omega \otimes a_i), \quad x_i \in \mathcal{M}, a_i \in A.$$

Lemma 5 *V is a unitary in $\mathcal{L}(\mathcal{H}_\omega \otimes A) = M(\mathcal{K}(\mathcal{H}_\omega)\otimes A)$ (the multipliers of the C^*-algebra $\mathcal{K}(\mathcal{H}_\omega)\otimes A$) and $\sigma(x) = V(x\otimes 1_A)V^*$, $x \in \mathcal{M}$.*

Proof. For any $\phi \in A^*, a,b \in A$, denote by $\phi(a\cdot b) \in A^*$ the linear functional $\phi(a\cdot b)(x) = \phi(axb)$, $x \in A$. The σ-invariance of ω yields :

$$\begin{aligned}&\phi((\omega\otimes id_A)((1_\mathcal{M}\otimes b^*)\sigma(x)(1_\mathcal{M}\otimes a)) = (\omega\otimes\phi(b^*\cdot a))(\sigma(x))\\ &\quad = \phi(b^*\cdot a)(1_A)\omega(x) = \phi(b^*a)\omega(x), \quad x\in\mathcal{M}, a,b\in A, \phi\in A^*,\end{aligned}$$

therefore we have for all $a,b \in A, x,y \in \mathcal{M}$:

$$\begin{aligned}&\langle V(x_\omega\otimes a), V(y_\omega\otimes b)\rangle_A = \langle\sigma(x)(1_\omega\otimes a), \sigma(y)(1_\omega\otimes b)\rangle_A\\ &\quad = (\omega\otimes id_A)((1_\mathcal{M}\otimes b^*)\sigma(y^*x)(1_\mathcal{M}\otimes a)) = \omega(y^*x)b^*a = \langle x_\omega\otimes a, y_\omega\otimes b\rangle_A\end{aligned}$$

and V follows isometry on $\mathcal{H}_\omega\otimes A$. Furthermore V is unitary since $\sigma(\mathcal{M})(1_\mathcal{M}\otimes A)$ is dense in $\mathcal{M}\otimes A$ and the relation $V(x\otimes 1_A) = \sigma(x)V$, $x\in\mathcal{M}$, is obvious. □

Definition 6 ([BS]) *A corepresentation of the Hopf C^*-algebra (A,Δ_A) is a unitary $V \in \mathcal{L}(\mathcal{H}_V\otimes A) = M(\mathcal{K}(\mathcal{H}_V)\otimes A)$ such that*

$$V_{12}V_{13} = (id_{\mathcal{L}(\mathcal{H}_V)}\otimes\Delta_A)(V).$$

All the corepresentations throughout this paper will be unitary unless specified otherwise. Note that in the case when $\dim\mathcal{H}_V < \infty$ V is called in [Wor] a (finite dimensional) representation of the quantum matrix pseudogroup $G = (A,u)$. Thus the representations of the quantum matrix pseudogroup $G = (A,u)$ are the corepresentations of (A,Δ_A) and we will call them simply the corepresentations of A.

Lemma 7 *The unitary V from Lemma 5 is a corepresentation of A.*