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F. COMETS

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Abstract. — We prove existence of a phase transition for the Sherrington-Kirkpatrick model at $\beta = 1$: making use of the domination by the spherical model, we derive a bound for the pressure as well as for the ground state energy.

1. Introduction. The Sherrington-Kirkpatrick (SK) model consists of a set of N binary spins $\sigma_i \in \{-1, +1\}$ with Hamiltonian

$$H_N(\sigma) = -\frac{1}{\sqrt{N}} \sum_{1 \le i < j \le N} J_{ij}\sigma_i\sigma_j - \frac{1}{\sqrt{2N}} \sum_{1 \le i \le N} J_{ii}\sigma_i^2 \tag{1.1}$$

where the couplings J_{ij} are independent Gaussian random variables with mean zero and unit variance. Compared with the original definition [SK] we have added the second summand in (1.1) which does not depend on σ in the binary case and does not change the results below. The partition function at inverse temperature β and the pressure of the model are the random variables

$$Z_N^{\mathrm{SK}} = \mathrm{E}_\sigma \exp\{-\beta H_N(\sigma)\}\,, \qquad p_N^{\mathrm{SK}} = \frac{1}{N} \log Z_N^{\mathrm{SK}} \tag{1.2}$$

depending on the J_{ij}'s; in (1.2) we have used the notation $\mathrm{E}_\sigma = 2^{-N} \sum_{\sigma \in \{-1,+1\}^N}$ to denote the average over the spin configurations, and we keep the symbol $\mathbb{E}$ for expectations over the couplings J_{ij}.

Among the few mathematical results for this model it is known that, at high temperature, the "quenched" and "annealed" behaviors coincide:

$$p_N^{\mathrm{SK}} \underset{N \to \infty}{\longrightarrow} \frac{\beta^2}{4} = \lim_{N \to \infty} \frac{1}{N} \log \mathbb{E}\, Z_N^{\mathrm{SK}}\,. \tag{1.3}$$

Convergence in probability and in L_p, $1 \le p < \infty$, follows from the fluctuations results of [ALR] or [CN], and convergence with probability one follows using in addition the concentration property proved in [T] from general considerations, or in [BGP] more directly. On the other hand, the convergence (1.3) cannot hold for large β since the entropy per spin is at most $\log 2$ for binary spins.

In this note we show that the convergence in (1.3) only happens when $\beta \leq 1$ and hence phase transition takes place at $\beta_c = 1$. For this we prove below a (weak) bound for the limit points of p_N^{SK}, which is a self-averaging quantity [PS]. This bound implies another one for the ground state energy.

These bounds reflect the domination of the SK model by the spherical model, where the uniform probability measure E_σ on the N-dimensional hypercube $\{-1,+1\}^N$ is replaced with the uniform probability measure E_η on the sphere

$$\{\eta \in \mathbb{R}^N \ ; \quad |\eta|^2 := \sum_{i=1}^{N} \eta_i^2 = N\} \ .$$

The partition function and pressure of this second model

$$Z_N^{\mathrm{S}} = \mathrm{E}_\eta \exp\{-\beta H_N(\eta)\} \ , \qquad p_N^{\mathrm{S}} = \frac{1}{N} \log Z_N^{\mathrm{S}} \tag{1.4}$$

depend only on the eigenvalues of the quadratic form H_N. Spherical models, introduced by Berlin and Kac [BK], are completely solvable in many instances. They are commonly analysed via the method of steepest descent, or more simply via the mean-spherical approximation (i.e., as Gaussian models where the spherical constraint is satisfied in the mean). The asymptotics of (1.4) are studied in [KTJ] and in [Th], but for completeness of this note, we give below a quick derivation of the bound.

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2. Spherical bound and consequences.

The $N \times N$ symmetric matrix M, with coefficients $M_{ij} = \frac{1}{2\sqrt{N}} J_{ij}$ if $i < j$ and $M_{ii} = \frac{1}{\sqrt{2N}} J_{ii}$, has a.s. simple eignevalues $\lambda_1 < \lambda_2 < \ldots < \lambda_N$ with normalized eigenvectors $\phi_1 = (\phi_{1,j})_{j \leq N}, \ldots, \phi_N$. Since the distribution of M is invariant under orthogonal transformations, the diagonal matrix Λ of the eigenvalues $\lambda_1, \ldots, \lambda_N$ is independent of the orthogonal matrix $\phi = (\phi_1, \ldots, \phi_N)$, and we may clearly choose the frame ϕ such that it is uniformly distributed on the set $O(N)$ of orthogonal matrices; in particular one has for any positive measurable function F

$$\mathbb{E}\{F(M)|\Lambda\} = \mathbb{E}_\phi \ F(\phi \Lambda \phi^t) \ , \qquad \mathbb{P}\text{-a.s.} \tag{2.1}$$

with $\mathbb{E}_\phi$ the expectation in ϕ uniformly distributed on $O(N)$. This implies that the spherical model dominates the SK model: more precisely, for any fixed binary spin σ, the distribution of the scalar products $(\sigma.\phi_i)_{i \leq N}$ under $\mathbb{E}_\phi$ is the uniform measure E_η on the sphere, and then

$$\mathbb{E}\{Z_N^{\mathrm{SK}}|\Lambda\} = Z_N^{\mathrm{S}} \ . \qquad \mathbb{P}\text{-a.s.} \tag{2.2}$$

On the other hand, by Wigner's semicircle law, the empirical measure of eigenvalues converges weakly [G]

$$\frac{1}{N}\sum_{k=1}^{N}\delta_{\lambda_k} \underset{N\to\infty}{\longrightarrow} w(\lambda)\,d\lambda := \frac{2}{\pi}(1-\lambda^2)^{\frac{1}{2}}\,\mathbb{1}_{]-1,1[}(\lambda)\,d\lambda \qquad \mathbb{P}\text{-a.s.} \qquad (2.3.a)$$

and moreover the maximal and minimal eigenvalues do not deviate [BY]

$$\lim_{N\to\infty}\lambda_N = -\lim_{N\to\infty}\lambda_1 = 1\,. \qquad \mathbb{P}\text{-a.s.} \qquad (2.3.b)$$

Let us now state our main result which implies non-analyticity of the pressure at $\beta = 1$.

Proposition: *a) For all $\beta \geq 1$,*

$$\limsup_{N\to\infty} p_N^{\mathrm{SK}} \leq \beta - \tfrac{1}{2}\log\beta - \tfrac{3}{4}\,. \qquad \mathbb{P}\text{-}a.s.$$

b) In particular, $\limsup_{N\to\infty} p_N^{\mathrm{SK}} < \beta^2/4$ $\mathbb{P}$-a.s. when $\beta > 1$, though the limit (1.3) *holds when* $0 \leq \beta \leq 1$.

In fact the bound in a) is equal to the limit of p_N^{S} [Th]. For $\beta > 1$ but close to 1, the bound is larger than the Sherrington-Kirkpatrick solution for the pressure of the SK model, and then it does not prove absence of self-averaging of the Edwards-Anderson order parameter [PS] for these values of β.

Proof. We will prove that

$$\limsup_{N\to\infty} p_N^{\mathrm{SK}} \leq \inf_{s>\beta}\left\{s - \tfrac{1}{2} - \tfrac{1}{2}\int \log 2(s-\beta\lambda)\,w(\lambda)\,d\lambda\right\} \qquad \mathbb{P}\text{-a.s.} \qquad (2.4)$$

Then, noticing that the function between braces in (2.4) is convex in s, and using the identities

$$\begin{cases}\int (s-\beta\lambda)^{-1}\,w(\lambda)\,d\lambda = 2\beta^{-2}\left[s-(s^2-\beta^2)^{\frac{1}{2}}\right] & (2.5.a)\\ \int \log(s-\beta\lambda)\,w(\lambda)\,d\lambda = \beta^{-2}s\left[s-(s^2-\beta^2)^{\frac{1}{2}}\right] + \log\left(\left[s+(s^2-\beta^2)^{\frac{1}{2}}\right]/2\right) - 1/2 & (2.5.b)\end{cases}$$

valid for $s > \beta$, it can be checked that the bound in (2.4) is achieved for $\beta < 1$ at the saddle-point $s = (\beta^2+1)/2$ and is equal to $\beta^2/4$, but corresponds to $s = \beta$ when $\beta \geq 1$ and is equal to $\beta - \frac{1}{2}\log\beta - \frac{3}{4}$.

We now prove (2.4). Let $s > s_0 > \beta$; and assume that $\beta\lambda_N \leq s_0$. Then,

$$a_N := \int_{\mathbb{R}^N} \exp\left\{\beta\zeta^t\Lambda\zeta - s|\zeta|^2\right\}d\zeta = \exp\left\{\tfrac{N}{2}\log 2\pi - \tfrac{1}{2}\log\det 2(sI-\beta\Lambda)\right\}. \qquad (2.6)$$

For $\epsilon \in\,]0,1[$, let $v_{N,\epsilon}$ be the Euclidean volume of $\left\{\zeta \in \mathbb{R}^N, |\zeta|^2/N \in [1-\epsilon, 1]\right\}$. The uniform probability measure on this domain makes the vectors $\eta = \sqrt{N}(\zeta_i/|\zeta|)_{i\leq N}$

and $|\zeta|^2/N$ independent, the first one with distribution $\mathbb{E}_\eta$ and the second one with mean $m_{N,\epsilon} \in [1-\epsilon, 1]$. Therefore, we have

$$a_N \geq \int \exp \frac{|\zeta|^2}{N}\{\beta\eta^t\Lambda\eta - sN\}\, \mathbb{1}_{N^{-1}|\zeta|^2 \in [1-\epsilon,1]}\, d\zeta\, v_{N,\epsilon}^{-1}.v_{N,\epsilon}$$
$$\geq v_{N,\epsilon}.\mathrm{E}_\eta \exp m_{N,\epsilon}\{\beta\eta^t\Lambda\eta - sN\}$$

from Jensen inequality for E_η. Since $s > s_0 \geq \beta\lambda_N$, the last term in braces is negative, the function $m \mapsto \mathrm{E}_\eta \exp m\{\beta\eta^t\Lambda\eta - sN\}$ is decreasing and

$$a_N \geq v_{N,\epsilon}\, \mathrm{E}_\eta \exp\{\beta\eta^t\Lambda\eta - sN\} = V_{N,\epsilon} \exp\{-Ns\}\, Z_N^{\mathrm{S}} .$$

Combining (2.6) and the estimate

$$v_{N,\epsilon} = \pi^{N/2}\Gamma(1+N/2)^{-1}\, N^{N/2}\left[1 - \exp\frac{N}{2}\log(1-\epsilon)\right]$$
$$= \exp\frac{N}{2}\{\log 2\pi + 1 + \mathcal{O}_\epsilon(1)\}$$

with some (deterministic) sequence $\mathcal{O}_\epsilon(1)$ going to zero, we obtain finally for $\beta\lambda_N \leq s_0$

$$p_N^{\mathrm{S}} \leq s - \tfrac{1}{2} - \frac{1}{2N}\sum_{i=1}^{N}\log 2(s - \beta\lambda_k) + \mathcal{O}_\epsilon(1) . \tag{2.7}$$

Letting $b = s - \frac{1}{2} - \frac{1}{2}\int \log 2(s-\beta\lambda)\, w(\lambda)\, d\lambda$ we define the following set A_N of couplings $A_N = \{\lambda_N \leq s_0/\beta, \lambda_1 > -2, s - \frac{1}{2} - \frac{1}{2N}\sum_{i=1}^N \log 2(s-\beta\lambda_k) < b+\epsilon\}$. We have

$$\begin{aligned}\mathbb{E}\, Z_N^{\mathrm{SK}}\, \mathbb{1}_{A_N} &= \mathbb{E}\, \mathbb{1}_{A_N} \mathrm{E}(Z_N^{\mathrm{SK}}|\Lambda) \\ &= \mathbb{E}\, \mathbb{1}_{A_N}\, Z_N^{\mathrm{S}} && \text{(from (2.2))} \\ &\leq \exp N(b+2\epsilon) && \text{for large } N \quad \text{(from (2.7))},\end{aligned}$$

$$\mathbb{P}\big(\{p_N^{\mathrm{SK}} \geq b+3\epsilon\} \cap A_N\big) \leq \exp -N\epsilon$$

and the Borel-Cantelli lemma implies that $\mathbb{P}(\limsup_{N\to\infty} \{p_N^{\mathrm{SK}} \geq b+3\epsilon\} \cap A_N) = 0$. On the other hand we know from (2.3.a and b) that $\mathbb{P}(\liminf_{N\to\infty} A_N) = 1$, therefore $\mathbb{P}(\limsup_{N\to\infty} \{p_N^{\mathrm{SK}} \geq b+3\epsilon\}) = 0$ for all $\epsilon > 0$. We have finally $\limsup_{N\to\infty} p_N^{\mathrm{SK}} \leq b$ $\mathbb{P}$-a.s., which shows (2.4).

The statement b) is an obvious consequence of a), except for $\lim_N p_N^{\mathrm{SK}} = \beta^2/4$ if $\beta = 1$; but this follows from convexity of p_N^{SK} in β, from (1.3) and a) for $\beta = 1$. See also [Gu] for the case $\beta = 1$. ∎