Astérisque

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Astérisque, tome 236 (1996), p. 125-131

<http://www.numdam.org/item?id=AST_1996__236__125_0>

Condenser Potentials

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Abstract. — Under appropriate hypotheses, the potential theory of a transient Markov process can be recovered from the condenser charges.

A central object in both the theory of Markov processes and in potential theory is the cone of excessive or superharmonic functions. This cone provides a critical link between the two subjects, and many important and useful theorems in Markov processes aim at a deeper understanding of the cone and its properties. Various subsets of this cone have been studied, also. For example, the collection of hitting probabilities proves to contain enough information to recover the potential theory of the process [5,6]. In this article, we suggest that another important link between Markov processes and potential theory is forged by the collection of condenser potentials and the associated collection of condenser charges. Condenser potentials receive little attention even in comprehensive tomes on potential theory. The condenser theorem for classical potential theory in $\mathbb{R}^d$ is the following and has a standard extension in the theory of Dirichlet spaces [9].

Theorem. *Let K and L be open sets with disjoint closures $\overline{K}$ and $\overline{L}$, and assume that $\overline{K}$ is compact. Then there exists a potential p of a signed measure μ such that:*
(i) $0 \le p \le 1$ a.e. on $\mathbb{R}^d$.
(ii) $p = 0$ a.e. on L and $p = 1$ a.e. on K.
(iii) The support of μ^+ is contained in $\overline{K}$ and the support of μ^- is contained in $\overline{L}$.

The potential p is in fact unique in $\mathbb{R}^d$ and uniqueness holds also in Dirichlet spaces.

We are aware of only one probabilistic study of condenser potentials, that being the 1977 note by K. L. Chung and R. K. Getoor [4]. They "guessed" that the condenser potential is simply the probability starting at x that Brownian motion hits K before it hits L. In their article, they deal with a Hunt process on a locally compact state space satisfying the duality assumptions in [1]. Throughout this article, we adopt the same assumptions: $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, P^x)$ is a Hunt process on a locally compact

Research of the first author supported in part by NSA grant MDA904–94–H–2052.

state space E satisfying the duality assumptions in Section VI–1 of [1]. In particular, $u(x,y)$ will denote the potential density and $U\gamma$ will denote the potential of a measure γ. For K and L Borel sets in E with disjoint closures, we define $p(x) = P^x[T_K < T_L]$ to be the condenser potential of the pair (K,L). Define $p_n = (P_K P_L)^n P_K 1$. Chung and Getoor's final result may be stated as follows.

Theorem. *If $\sum_n p_n$ converges, then $p = U\mu$ is the condenser potential of (K,L), where the measure μ is obtained as follows:*

(i) μ^+ is the capacitary measure μ_K of K relative to (X, T_L), the process X killed when it hits L.

(ii) $\mu^- = \widehat{P}_L \mu_K$ is the co–balayage onto L of μ_K.

Chung and Getoor also investigate several conditions guaranteeing $\sum_n p_n$ converges.

In this article, we investigate what rôle the condenser potential p and the condenser measure μ can play in potential theory, now that they have interesting probabilistic interpretations. We begin by characterizing the condenser potential in the symmetric case in a style akin to Hunt's balayage theorem. Our real interest in this article lies in studying the non–symmetric case, so we present this result in passing to help the reader with intuition about condenser potentials. Let $\mathcal{K}$ denote the collection of probability measures on $\overline{K}$, and let $\mathcal{L}$ denote the collection of positive measures on $\overline{L}$. Recall that the mutual energy of two measures λ and ρ is defined by

$$\ll \lambda, \rho \gg = \int_E \int_E u(x,y)\lambda(dx)\rho(dy)$$

It is by now a standard exercise to extend this to be an inner product on the space of signed measures π with $\ll |\pi|, |\pi| \gg < \infty$. This space is a pre–Hilbert space. We shall denote the norm of π by $||\pi||$.

Theorem. *Assume that $u(x,y) = u(y,x)$. Assume also that every potential $U\gamma$ of a measure γ is lower semicontinuous on E and continuous off the support of γ. Let $\overline{K}$ and $\overline{L}$ be compact and disjoint. The unique measure $\gamma - \nu$ which minimizes*

$$\inf_{\gamma \in \mathcal{K}, \nu \in \mathcal{L}} ||\gamma - \nu||$$

is a constant times the condenser measure μ.

Proof. We show first that the inf above is achieved by measures $\rho \in \mathcal{K}$ and $P_L\rho \in \mathcal{L}$. Choose $(\rho_n) \subset \mathcal{K}$ and $(\pi_n) \subset \mathcal{L}$ such that the sequence $||\rho_n - \pi_n||$ converges to $e = \inf_{\gamma\in\mathcal{K},\nu\in\mathcal{L}} ||\gamma - \nu||$. Then, by reducing to a subsequence if necessary, ρ_n converges weakly to a measure $\rho \in \mathcal{K}$ and π_n converges weakly to a measure $\pi \in \mathcal{L}$. A standard Dirichlet space computation yields $||\rho - \pi|| \leq \liminf_{n\to\infty} ||\rho_n - \pi_n|| = e$. But $||\rho - \pi|| \geq ||\rho - P_L\rho||$ since $P_L\rho$ is the unique measure in $\mathcal{L}$ minimizing the distance between ρ and $\mathcal{L}$. Thus $e = ||\rho - P_L\rho||$ with $\rho \in \mathcal{K}$ and $P_L\rho \in \mathcal{L}$.

Now let $\xi \in \mathcal{K}$ and $\beta \in \mathcal{L}$ be any pair of measures such that $||\xi - \beta|| = e$. Note that $\beta = P_L\xi$ since $P_L\xi$ is the unique measure in $\mathcal{L}$ minimizing the distance between ξ and $\mathcal{L}$. Take another measure $\lambda \in \mathcal{K}$ of finite energy, and let $t > 0$. Then

$$||t(\lambda - \xi) + \xi - \beta|| = ||(1-t)\xi + t\lambda - \beta|| \geq ||\xi - \beta||$$

Thus

$$t^2||\lambda-\xi||^2+2t \ll \lambda-\xi,\xi-\beta \gg \geq 0$$

for every $t>0$, and we conclude that

$$\ll \lambda-\xi,\xi-\beta \gg \geq 0$$

That is,

$$\int U(\xi-\beta)d(\lambda-\xi)\geq 0$$

so

$$\int U(\xi-\beta)d\lambda \geq \int U(\xi-\beta)d\xi = \int U(\xi-\beta)d(\xi-\beta)=e^2$$

(the first equality holding since $U(\xi-\beta)=0$ on L). Since $\lambda\in\mathcal{K}$ is arbitrary, $U(\xi-\beta)\geq e^2$ on K, except perhaps on a set of capacity zero. Taking $\lambda=\xi$, we get

$$\int U(\xi-\beta)d\xi = \int U(\xi-\beta)d(\xi-\beta)=e^2$$

so $U(\xi-\beta)=e^2$ a.e. ξ on K. Since $U\xi\leq U\beta+e^2$ a.e. ξ, we have $U\xi\leq U\beta+e^2$ on E by the maximum principle. To summarize, $U(\xi-\beta)\leq e^2$, $U(\xi-\beta)=e^2$ a.e. ξ on K, $\beta=P_L\xi$ and $U(\xi-\beta)=0$ on L. By uniqueness of condenser potentials, $\xi=\rho$ and $\beta=P_L\rho$. □

In [7], Glover, Hansen and Rao observed that the potential theory of a symmetric process can be reconstructed from the capacities. This result can also be found in Choquet [3], at least in the case where points are polar. In the case where the process hits points, Glover, Hansen and Rao proved the following formula, which will be useful for the purposes of comparison later.

Theorem. *Assume that $u(z,w)=u(w,z)$ and $P^z(T_{\{z\}}<\infty)>0$ for all z and w in E. Fix x and y in E, let a be the capacity of $\{x\}$, let b be the capacity of $\{y\}$, and let c be the capacity of the set $\{x,y\}$. Then*

$$u(x,y)=\frac{1-\sqrt{1-c\left(\dfrac{1}{a}+\dfrac{1}{b}-\dfrac{c}{ab}\right)}}{c}$$

If $a=b$, then

$$u(x,y)=\frac{2}{c}-\frac{1}{a}$$

Symmetry is needed in the theorem above: one cannot recover the potential theory of a nonsymmetric process from the capacities, in general. However, one can recover neatly the potential theory of a nonsymmetric process from the condenser charges, as follows.

Definition. *Let $U\mu$ be the condenser potential of the sets (K,L). The condenser charge associated with the sets (K,L) is defined to be $\mu(E)$, and will be denoted by $c(K,L)$.*

Under the hypotheses of Chung and Getoor's theorem, $c(K,L) > 0$, and, for fixed L, the map $K \to c(K,L)$ is a capacity which is alternating of order infinity, since it is simply the capacity of the process X killed the first time it hits L. We will need some notation to state the next result. For $x \neq y$, let $c_{xy} = c(\{x\},\{y\})$ and $c_{yx} = c(\{y\},\{x\})$, let $a = c(\{x\},\emptyset)$ and $b = c(\{y\},\emptyset)$.

Theorem. *Assume that $P^z(T_{\{z\}} = 0) = 1$ for all $z \in E$. Then $u(x,x) = a^{-1}$.*

- *If $c_{xy} = 0$, then $u(y,x) = b^{-1}$ and $u(x,y) = 0$.*
- *If $c_{yx} = 0$, then $u(x,y) = a^{-1}$ and $u(y,x) = 0$.*
- *If $c_{xy} \neq 0$ and $c_{yx} \neq 0$, then*

$$u(y,x) = \frac{1}{c_{yx}} - \frac{c_{xy}}{ac_{yx}}$$
$$u(x,y) = \frac{1}{c_{xy}} - \frac{c_{yx}}{bc_{xy}}$$

Proof. $u_{xx} = a^{-1}$, so u_{xx} can be determined from condenser charges. If $x \neq y$, let $\mu = \mu_{\{x\},\{y\}}$, $\mu_x = \mu(\{x\})$ and $\mu_y = \mu(\{y\})$. Then

$$\begin{aligned} u_{xx}\mu_x + u_{xy}\mu_y &= 1 \\ u_{yx}\mu_x + u_{yy}\mu_y &= 0 \end{aligned} \tag{1}$$

If the determinant $D = u_{xx}u_{yy} - u_{xy}u_{yx} = 0$, then $u_{xx} = u_{yy} = u_{xy} = u_{yx}$ since the maximum principle guarantees $u_{xx} \geq u_{xy}$, $u_{yy} \geq u_{yx}$, $u_{xx} \geq u_{yx}$, and $u_{yy} \geq u_{xy}$. But this would imply that the restriction of u to $\{x,y\} \times \{x,y\}$ is not the potential density of a transient two–state Markov chain, which would be a contradiction. So $D > 0$. Then $\mu_x = u_{yy}/D$ and $\mu_y = -u_{yx}/D$, so $c_{xy} = (u_{yy} - u_{yx})/D$. Similarly, $c_{yx} = (u_{xx} - u_{xy})/D$. If $c_{xy} = 0$, then $u_{yx} = u_{yy} = b^{-1}$ and $c_{yx} = b$. By Chung and Getoor's result [4], $c_{yx} = b - b\widehat{P}_{\{x\}}(y,\{x\})$, and we conclude that $\widehat{P}_{\{x\}}(y,\cdot) = 0$. By time reversal, it follows that $P^x(T_y < \infty) = 0$ which implies $u(x,y) = 0$.

If $c_{xy} \neq 0$ and $c_{yx} \neq 0$, then $(u_{yy} - u_{yx})/c_{xy} = (u_{xx} - u_{xy})/c_{yx}$. Solving for u_{xy} and substituting into equation (1), we obtain a quadratic in u_{yx} with solution

$$u_{yx} = \frac{\left(1 + \frac{c_{yx}}{b} - \frac{c_{xy}}{a} \pm \sqrt{\left(\frac{c_{xy}}{a} - 1 - \frac{c_{yx}}{b}\right)^2 - 4c_{yx}\left(\frac{1}{b} - \frac{c_{xy}}{ab}\right)}\right)}{2c_{yx}}$$
$$= \frac{1 + \frac{c_{yx}}{b} - \frac{c_{xy}}{a} \pm \left|1 - \left(\frac{c_{xy}}{a} + \frac{c_{yx}}{b}\right)\right|}{2c_{yx}}$$