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Abstract. — We study a certain family of typed branching diffusions where the type of each particle moves as an Ornstein-Uhlenbeck process and binary branching occurs at a rate quadratic in the particle's type. We calculate the 'left-most' particle speed for the branching process explicitly, aided by close connections with harmonic oscillator theory. The behaviour of the system changes markedly below a certain critical temperature parameter.

In the high-temperature regime, the study of various 'additive' martingales and their use in a change of measure method provides the proof of the almost sure speed of spread of the particle system.

Also, we briefly mention how to use the martingale results of the branching diffusion model in representations of travelling-wave solutions for the associated reaction-diffusion equation.

1. Introduction

Our aim is to produce a series of papers on a certain family of typed branching diffusions each with rich structure. The present paper introduces the simplest (binary-branching) model and (except for a 'sneak preview' of the critical-temperature phase in the Section 9) studies this model only in the high-temperature phase in which there is a high degree of ergodicity. Here, many standard methods are applicable, though we have been able to carry them through only because the model's close relation to the harmonic oscillator allows explicit calculations; the first calculations also have a long history in probability going back to Cameron and Martin – see Sections 5.13-5.15 of Itô and McKean (1965). Some of the calculations necessary for our approach are rather complicated; and these are only sketched here – see Harris (1995) and Harris and Williams (1995) for more details. We deal with the substantive points of rigour, but skip some details of rigour to keep the text to an appropriate length.

We begin by recalling how certain 'linear' expectations for the branching process may be calculated by considering a one-particle system, and we derive certain martingale properties. We then study in some detail the large-deviation heuristics for the problem, emphasizing the rôle of Legendre transformations. Next, we use a method of Neveu to establish uniform-integrability properties of certain martingales; this requires calculation of an expectation which reflects the *non*-linearity of the system, and we are saved only because Meyer's *opérateur carré du champ* behaves well. By exploiting a change-of-measure technique ('exponential tilting' in the exotic/quixotic terminology of statisticians), we *prove* the results suggested by large-deviation theory. The martingale methods have the significant bonus that they imply existence of monotone travelling waves associated with the model. In the present context, it may not be easy to establish the existence of these waves by analysis. Neveu's method of proving *lack* of uniform integrability of certain martingales will be important for proving uniqueness in some cases, non-existence in others, for monotone travelling waves. This idea is developed in full for a simpler problem in Champneys, Harris, Toland, Warren and Williams (1995); in the present context, it requires difficult *a priori* estimates. We also refer the reader to the Champneys *et al* paper for a list of references to which the present paper is equally indebted.

Further study of the high-temperature regime is made in Harris (1995), and will be continued in other joint papers. The changes of measure have some bizarre features which we wish to discuss further, bringing in important ideas from Chauvin and Rouault (1988, 1990). The long-term behaviour of the 'Gibbs-Boltzmann' measure $J_\lambda(t)$ which assigns mass $J_\lambda(t,k)$ as at (6.1) to the point $(X_k(t), Y_k(t))$ is the most fascinating aspect of the high-temperature phase. Note that the fundamental martingale $Z_\lambda^-(t)$ gives the 'partition function'. The study of the long-term behaviour of J_λ is closely related to that of the 'excited-state' martingales for our system.

A major challenge for the binary-branching model is the low-temperature regime ($\theta < 8r$) in which all of the methods used here fail: the expected number of particles in a region blows up, though the number of particles remains almost surely finite. Other models present other challenges.

2. The Branching Model

We consider a typed branching diffusion where, for time $t \geq 0$,

$N(t)$ is the number of particles alive,
$X_k(t)$ in $\mathbb{R}$ is the spatial position of the k^{th}-born particle,
$Y_k(t)$ in $\mathbb{R}$ is the 'type' of the k^{th}-born particle,

$\big(N(t); X_1(t), \ldots, X_{N(t)}; Y_1(t), \ldots, Y_{N(t)}\big)$ is the current state of the particle system.

The *type* moves on the real line as an Ornstein-Uhlenbeck process associated with the differential operator (generator)

$$Q_\theta := \frac{\theta}{2}\left(\frac{\partial^2}{\partial y^2} - y\frac{\partial}{\partial y}\right)$$

where θ is a positive real parameter considered as the *temperature* of the system. The *spatial* motion of a particle of type y is a driftless Brownian motion with variance

$$A(y) := ay^2, \qquad \text{where } a \geq 0.$$

The *breeding* of a type y particle occurs at a rate

$$R(y) := ry^2 + \rho, \qquad \text{where } r, \rho \geq 0,$$

and we have one child born at these times (binary splitting). A child inherits its parent's current type and (spatial) position then moves off *independently* of all others. Particles live forever (once born!).

Let $\mathbb{P}^{x,y}$ and $\mathbb{E}^{x,y}$ represent probability and expectation when the process starts from $(N; \mathbf{X}, \mathbf{Y}) = (1; x; y)$.

For starting point $(N; \mathbf{X}; \mathbf{Y}) = (1; 0; 0)$, we have

$$\mathbb{P}^{0,0}(N(t) = 1 \,|\, \sigma(Y_1(s) : s \leq t)) = \exp\left(-\int_0^t R(Y_1(s))\, ds\right),$$

and on the set $\{N(t) \geq k\}$ we have

$$\begin{aligned}(2.1) \qquad &\mathbb{P}^{0,0}\Big(X_k(t) \in F \,\Big|\, \sigma\big(N(s), \mathbf{Y}(s) : s \leq t\big)\Big) \\ &= \int_F \left\{2\pi \int_0^t A\big(Y_k(s)\big)\, ds\right\}^{-\frac{1}{2}} \exp\left(-\frac{x^2}{2\int_0^t A\big(Y_k(s)\big)\, ds}\right) dx,\end{aligned}$$

where $Y_k(s)$ is the type of the unique 'ancestor' alive at s of the k-th particle alive at time t.

We are going to consider r, ρ, a as fixed, and look at the effects of changing the temperature θ. One of our main concerns is: what is 'the velocity of the leftmost particle'; to be precise, what is the value of

$$\text{Vel} := \lim_{t \to \infty} L(t)/t$$

(we prove that the almost sure limit does exist), where

$$L(t) := \inf_{1 \leq k \leq N(t)} X_k(t)?$$

The temperature controls the balance of competition between the ergodic mixing of the Ornstein-Uhlenbeck process (which increases with θ) and the large breeding rate and large diffusion coefficient for the X-motion away from the type-origin. This is reflected in the answer

$$\text{Vel} = -\tilde{c}(\theta),$$

where

$$\tilde{c}(\theta)^2 = \begin{cases} 2a\left(r+\rho+\dfrac{2(2r+\rho)^2}{\theta-8r}\right) & \text{for } \theta > 8r, \\ +\infty & \text{for } \theta \leq 8r. \end{cases} \tag{2.2}$$

When θ is very large, the system may be approximated by a 'mean field' model in which $A(Y)$ is replaced by its mean a and $R(Y)$ by its mean $r+\rho$ under the (standard normal) invariant law of the type process.

In all but the last section of this paper,

we assume that $\theta > 8r$.

The challenging low-temperature cases and many other things are left to other occasions.

3. Calculations using the One-Particle System

Let (ξ, η) be a process behaving like a single particle's space and type motions in the branching model described above. Thus, ξ is a Brownian motion controlled by an Ornstein-Uhlenbeck process η, and (ξ, η) has formal generator $\mathcal{H}$, where

$$(\mathcal{H}F)(x,y) = \frac{1}{2}A(y)\frac{\partial^2 F}{\partial x^2} + (\mathcal{Q}_\theta F)(x,y) = \frac{1}{2}A(y)\frac{\partial^2 F}{\partial x^2} + \frac{\theta}{2}\left(\frac{\partial^2 F}{\partial y^2} - y\frac{\partial F}{\partial y}\right).$$

Of course, η is an autonomous Markov process with generator $\mathcal{Q}_\theta$ and with (standard normal) invariant density

$$\phi(y) := (2\pi)^{-\frac{1}{2}}\exp(-\tfrac{1}{2}y^2).$$

For functions h_1, h_2 on $\mathbb{R}$, we define the $L^2(\phi)$ inner product:

$$\langle h_1, h_2\rangle_\phi := \int_{\mathbb{R}} h_1(y)h_2(y)\phi(y)dy.$$

The following principle is used repeatedly.

(3.1) LEMMA: 'From One to Many'. *For any non-negative Borel function f on $\mathbb{R}\times\mathbb{R}$, we have*

$$\mathbb{E}^{x,y}\left(\sum_{k=1}^{N(t)} f\big(X_k(t), Y_k(t)\big)\right) = \mathbb{E}^{x,y}\left(\exp\left(\int_0^t R(\eta_s)\,ds\right) f(\xi_t, \eta_t)\right).$$

This principle is often combined with a change-of-measure formula for Ornstein-Uhlenbeck processes. We use $\mathrm{OU}(\theta, \mu)$ to represent an Ornstein-Uhlenbeck process with variance θ and drift parameter μ, thus with generator $\frac{\theta}{2}\frac{\partial^2}{\partial y^2} - \mu y\frac{\partial}{\partial y}$.