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Abstract. — Let $X(t)$, $0 \leq t \leq 1$, be a bridge from 0 to 0 with exchangeable increments on $D[0,1]$. We obtain the n.a.s.c. for the sojourn below 0 to be uniformly distributed, or equivalently for X to have a uniform index of the (unique) supremum. This is applied to Lévy bridges.

It seems particularly fitting for the present author to be given an opportunity to contribute to a volume in honor of Meyer and Neveu. Professor Meyer alone, over the years, has rewritten, revised, and expanded not fewer than five of our research papers, mostly as part of his herculean efforts on behalf of the Seminaire de Probabilités. There are various anecdotes concerning these papers which, if space permitted, we would gladly include. However, it seems fair to say that Meyer always put business before amusement, and following his lead we must be content to do likewise. Suffice it to say that both the subject and the author are lastingly indebted for his contributions. The present paper, however, is already indebted to a referee, so we can hope that it, at least, will not merit his revision.

In his famous paper [8], P. Lévy obtained the arcsine law for the positive sojourn of Brownian motion, and also the uniform law for the positive sojourn of Brownian bridge. Very recently ([5]) R. K. Getoor and M. J. Sharpe have obtained the necessary and sufficient conditions for the same arcsine law to hold for a diffuse Lévy process X on R. One purpose of the present paper is to do the analogous thing (but without the "diffuse" assumption) for the uniform law, at least if we understand by "bridge" the process $X_t - tX_1$, $t \leq 1$.

Also in the paper [8], Lévy obtained the arcsine law for the distribution of the last exit time g from 0 before $t = 1$. Since Lévy knew that $M(t) - B(t) \stackrel{d}{=} |B(t)|$, where $M(t) = \max_{s \leq t} B(s)$ ($B(s)$ being a Brownian motion) it followed immediately (although he does not mention it) that the location (abscissa) of the maximum of B in $0 < t < 1$ again has the arcsine law. He also probably realized that the location of the maximum of the bridge B^0 is uniformly distributed.

Both of these facts extend to processes with exchangeable increments whenever the corresponding laws for the positive sojourn are valid, by virtue of the identity that the law of the positive sojourn is the same as that of the location of the first supremum in [0,1]. This identity has a combinatorial basis in the analogous discrete parameter case, due to E. Sparre-Andersen. It was extended by a limit procedure to Lévy processes by Pecherskii and Rogozin [14], and to Lévy bridges by J. Bertoin [12]. In the present paper it is extended to processes with exchangeable increments (Theorem 1.4*). We wish to thank a referee for sketching this proof, based on the discrete parameter case (see Theorem 2 of W. Feller [13, XII. 8] for this case). However since this is a rather hard result, and the others seem much more intuitive, we have indicated it and the results depending on it with an asterisk.

For diffuse Lévy processes, the necessary and sufficient condition for the arcsine law of positive sojourn is $P\{X_t > 0\} = \frac{1}{2}, t > 0$. By contrast, for diffuse Lévy **bridges** the uniform law of positive sojourn **always** holds. In both cases the surprising level of generality goes back to Sparre-Andersen's work in the discrete parameter setting [1,2]. Indeed, a formula of [2] is used in [5]. Our debt is less concrete, although our reasoning is already implicit in [1]. It seems that for bridges the set-up of a discrete parameter, as in [1], only obscures the relative simplicity of the continuous parameter case.

Both the uniform sojourn law and the uniform location of the maximum are first obtained, in Section 1, for processes with exchangeable increments, where we rely on a representation given in O. Kallenberg [6]. Here it seems natural to replace the notion of bridge by the process linearly centered to vanish at $t = 1$. For Lévy processes, however, usage favors using the term "bridge" for a process <u>conditioned</u> to vanish at $t = 1$. Accordingly, we treat the two concepts separately in Section 2, although, generally speaking, the same uniform laws hold for both. In fact the two concepts coincide only in the Gaussian case (Theorem 2.2), and the definition by conditioning of course requires some supplementary hypothesis. We have found Condition (C) of Kallenberg [7] to be most adaptable to our needs at this point (but see the Remarks after Lemma 2.6).

Section 1. The uniform law for linearly centered processes with exchangeable increments.

A certain part of the theorems we wish to prove can be formulated for an arbitrary measurable function $f(t)$, $0 \leq t < 1$. We set $S(x, f) = \int_0^1 I_{(-\infty,x]}(f(t))dt$, $-\infty < x < \infty$. Noting that $\lim_{x\to-\infty} S(x, f) = 0$, $\lim_{x\to+\infty} S(x, f) = 1$, and $S(x, f)$ is non-decreasing and continuous to the right, we call $S(x, f)$ the "sojourn distribution function" of f. More generally, if $X_t(w)$ is a measurable stochastic process, $0 \leq t < 1$, we call $S(x, X(w))$ the (random) sojourn distribution of X, and when X is understood from context we abbreviate to simply $S(x)$. In that case, it is clear that $S(x)$ is a stochastic process associated with X. We say that f (or X) has continuous sojourn distribution if $S(x, f)$ (or $S(x, X)$, P-a.s.) is continuous in x. Now a critical result for the sequel is

Lemma 1.1(a). *Let f have continuous sojourn distribution, and let U be a uniformly distributed random variable on (0,1). Let*

$$X(t,w) = f((t+U)\mathrm{mod}1) - f(U) = \begin{cases} f(t+U) - f(U); & t < 1-U \\ f(t+U-1) - f(U); & 1-U \leq t < 1 \end{cases}$$

Then $P\{S(0,X) \leq x\} = x$, $0 < x < 1$, that is $S(0,X)$ has the same law as U.

Proof. Since $S(x,f)$ is continuous, for $0 < p < 1$ there is a number x_p for which $S(x_p,f) = p$. Then if $f(t) < x_p$ we have

$$\begin{aligned} &\int_0^1 I_{(-\infty,0]}(f((t+s)\mathrm{mod}1) - f(t))ds \\ &\leq \int_0^1 I_{(-\infty,0]}(f((t+s)\mathrm{mod}1) - x_p)ds \\ &= \int_0^1 I_{(-\infty,x_p]}f((t+s)\mathrm{mod}1)ds \\ &= p. \end{aligned}$$

Similarly, if $f(t) > x_p$, then

$$\begin{aligned} &\int_0^1 I_{(-\infty,0]}(f((t+s)\mathrm{mod}1) - f(t))ds \\ &\geq \int_0^1 I_{(-\infty,x_p]}f((t+s)\mathrm{mod}1)ds \\ &= p. \end{aligned}$$

Thus we have

$$\begin{aligned} &S(0,X) \leq p \quad \text{if} \quad f(U) < x_p, \quad \text{and} \\ &S(0,X) \geq p \quad \text{if} \quad f(U) > x_p. \end{aligned}$$

Now $P\{f(U) \leq x_p\} = p$, and since f has continuous sojourn distribution,

$$P\{f(U) = x_p\} = \int_0^1 I_{\{x_p\}}(f(s))ds = 0.$$

So it follows that $P\{S(0,X) \leq p\} \geq p$ and $P\{S(0,X) \geq p\} \geq 1-p$. By addition we get $P\{S(0,X) = p\} = 0$, and finally $P\{S(0,X) \leq p\} = p$, as asserted.

The second appearance of the uniform law which we intend to treat concerns the location, or argument, of the supremum. Here we shall assume that all functions or processes considered are right-continuous and left limited, so that their paths are in the space D [0, 1]. In the case of processes, we use the coordinate filtration, augmented by all P-null sets. This is equivalent to the augmented topological filtration of the complete separable metric space–see [3] for more details. This approach has the advantage that the supremum and the essential supremum coincide, so we need not treat them separately.

For $f \in D\,[0,\,1]$, we adjust the definition at 1 by setting $f(1) = f(1-)$, and we define $f(0-) = f(1)$, so that f can be viewed as defined on a circle. Let $Mf = \sup_{0 \le t < 1} f(t)$. We say that f has unique location of supremum (or just unique supremum) if there is a unique t_0, $0 \le t_0 < 1$, with $Mf = f(t_0-) \vee f(t_0)$, and we write $t_0 = ArgMaxf = AMf$. If this holds $P - a.s.$ for a process X, we write $AM(X)$ for its location (set=0 where not unique). We note that for any f there exists at least one t_0 with $Mf = f(t_0-) \vee f(t_0)$, so there is no problem as to existence.

Lemma 1.1(b). *If f has unique supremum, and $X = X(t,w)$ is as in Lemma 1.1(a), then $AM(X)$ has the same law as U.*

Proof. If $t_0 = AMf$, then one sees that $AM(X) = (1 + t_0 - U)(\text{mod}1)$, so the result follows.

We will apply this to certain processes with exchangeable increments. From now on, all processes considered will be assumed to have paths $X(\cdot,w) \in D[0,1]$, where $D[0,1]$ is the measurable space of right-continuous, left-limited real-valued functions (see [3] for details). We recall ([6]) that X_t has exchangeable increments if, for each n, the joint law of $\left\{X\left(\frac{k}{n}\right) - X\left(\frac{k-1}{n}\right); 1 \le k \le n\right\}$ is that same as that of $\left\{X\left(\frac{\sigma(k)}{n}\right) - X\left(\frac{\sigma(k)-1}{n}\right); 1 \le k \le n\right\}$ for every permutation σ of $\{1,2,\dots,n\}$. We will need to use the

Representation Theorem. *(Kallenberg, [6]). The process X_t, $X_0 = 0$, has exchangeable increments if and only if it may be represented in the form*

$$X_t = \alpha t + \sigma B_o(t) + \sum_{j=1}^{\infty} \beta_j (1(t - t_j) - t); \quad 1(s) \doteq \begin{cases} 0; & s < 0 \\ 1; & s \ge 0 \end{cases}, \tag{1.1}$$

where

(a) $B_o(t)$ is a Brownian bridge, $0 \le t \le 1$,

(b) α, σ and $\beta_1, \beta_2, \dots$ are real-valued random variables (on the probability space of X), independent of $B_o(\cdot)$, $0 \le \sigma$, and $\sum_{j=1}^{\infty} \beta_j^2 < \infty$, P-a.s.

(c) t_j, $1 \le j$, are uniformly distributed on (0, 1), independent of each other and of the random variables in (a) and (b).

<u>Remark.</u> Any or all of the variables in (a) and (b) may assume the value 0. The series, if infinite, converges a.s. uniformly in $t \le 1$.

Given such a process X_t, we set $Y_t \doteq X_t - tX_1$, $0 \le t < 1$. The following Lemma is the key to applying Lemma 1.1(a) to Y_t.