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G. LETAC

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The Function $\exp[-p \operatorname{Trace}\sqrt{2A}]$ as a Laplace Transform on Symmetric Matrices

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Abstract. — This note shows that if $p > 0$ and if S_+ is the set of symmetric positive definite matrices, then the function on S_+ defined by $A \mapsto \exp(-\operatorname{Trace} p\sqrt{2A})$ is the Laplace transform of a non positive function concentrated on S_+ if $n \geq 2$. This function is explicitely computed for $n = 2$. This computation is generalized to a Lorentz cone. The link of this question with the inverse Gaussian distributions in probability theory is also discussed, as well as the general problem of considering $\det L(A)$ as a Laplace transform on symmetric matrices when $L(\lambda)$ is a Laplace transform on the real line.

§1. Introduction. For $p > 0$, define the stable probability distribution of order 1/2 on $\mathbb{R}^+$:

$$\mu_p(dx) = \frac{p}{\sqrt{2\pi}} x^{-\frac{3}{2}} \exp\left(-\frac{p^2}{2x}\right) \mathbb{1}_{(0,+\infty)}(x)\, dx \tag{1.1}$$

Then its Laplace transform, evaluated at $\lambda > 0$, is :

$$\int_0^\infty \exp(-\lambda x)\, \mu_p(dx) = \exp(-p\sqrt{2\lambda}) \tag{1.2}$$

(See e.g. Feller 1970, p. 436 (3.4)).

Probability distributions (1.1) can be imbedded in the three parameter family of the so called "generalized inverse Gaussian distributions" defined for (a, b, λ) in $(0, +\infty) \times [0, +\infty) \times \mathbb{R}$ by

$$\mu_{\lambda,a,b}(dx) = \left(K_\lambda(\sqrt{ab})\right)^{-1} a^{\frac{\lambda}{2}} b^{-\frac{\lambda}{2}} x^{\lambda-1} \exp -\tfrac{1}{2}(ax + bx^{-1})\, \mathbb{1}_{(0,+\infty)}(x)\, dx \tag{1.3}$$,

where K_λ is a Bessel function (Watson 1966, p. 91).

Probability distributions (1.3) have a natural extension to the space of symmetric (n, n) real matrices, which extends nicely the fact that (1.3) is the distribution of a random continued fraction whose coefficients are independent and gamma distributed (see Letac and Seshadri 1983). This extension has been performed by E. Bernadac (1992) and even been made on general symmetric cones (see Bernadac 1993 and 1995). In this extension, the gamma distributions are replaced by the Wishart distributions on symmetric real matrices or on symmetric cones.

However, in this extension, the particular role played by $\lambda = -1/2$ when specializing (1.3) to (1.1) disappears, and although the extension of (1.3) to matrices is natural, extension of (1.1) is not. So one can look for an other path, and instead of trying to generalize (1.3), through for instance continued fractions, one can try to generalize (1.1) to symmetric matrices through (1.2). To describe what we have in mind, it is better to introduce a few definitions now.

Let E be a Euclidean space with dimension n, and let S be the space of symmetric endomorphisms of E. We equip S also with a Euclidean structure through the scalar product on S

$$(a, b) \mapsto \frac{1}{n} \operatorname{Trace} ab .$$

If $I \subset \mathbb{R}$, one denotes by $S(I)$ the set of a in S with eigenvalues in I; $S(I)$ is convex if I is an interval. For simplicity we write $S_+ = S((0, +\infty))$ (resp. $\bar{S}_+ = S([0, +\infty)))$ the cone of symmetric positive-definite (resp. positive) endomorphisms. Also, if e is a basis in E and a is in S, we write $[a]_e$ as its matrix in base e.

Let $f : I \to \mathbb{R}$ be any function. Suppose that a is in $S(I)$ and that e is an orthonormal basis which diagonalizes a, with $[a]_e = \operatorname{Diag}(\lambda_1, \ldots, \lambda_n)$. Then it is a standard exercise to show that $\tilde{f}(a)$ in S defined by

$$[\tilde{f}(a)]_e = \operatorname{Diag}\left(f(\lambda_1), \ldots, f(\lambda_n)\right) \tag{1.4}$$

actually does not depend on e. Thus $\tilde{f} : S(I) \mapsto S$ is a well defined function. Furthermore, if I is an interval and if the derivative f' exists on I, then $\tilde{f}$ is differentiable, and its differential $(\tilde{f})'(a)$ on a, evaluated at the point h of S, is computed as follows : defining $g : I \times I \to \mathbb{R}$ by :

$$g(\lambda, \lambda) = f'(\lambda) \qquad \text{and} \qquad g(\lambda, \mu) = \big(g(\lambda) - g(\mu)\big)/(\lambda - \mu) \quad \text{if} \quad \lambda \neq \mu$$

then, if e is an orthonormal basis with $[a]_e = \operatorname{Diag}(\lambda_1, \ldots, \lambda_n)$, we have

$$\left[(\tilde{f})'(a)(h)\right]_e = \big(g(\lambda_i, \lambda_j) h_{ij}\big) , \qquad \text{for} \quad [h]_e = (h_{ij}) . \tag{1.5}$$

The proof of (1.5) is a not so easy exercise in advanced calculus.

From (1.5), one deduces two facts. Asume that I is an open interval, and consider the function

$$a \longmapsto \operatorname{Trace} \tilde{f}(a) \qquad S(I) \longrightarrow \mathbb{R} \tag{1.6}$$

Then if f' exists, the differential of (1.6) in a is $(\tilde{f'})(a)$, from (1.5) (Note that we identify S with its dual through the Euclidean structure of S, and the differential of a real function on S can then be called a gradient). Furthermore, assume that f is convex on I. Then (1.6) will be convex on $S(I)$: to see this point, assume that f'' exists. Then, for arbitrary u in S and a in $S(I)$ (which is an open convex subset of S), there exists $\alpha > 0$ such that the function

$$(-\alpha, \alpha) \longrightarrow \mathbb{R} \qquad t \longmapsto F(t) = \operatorname{Trace} \tilde{f}(a+tu)$$

is well defined. With the help of (1.5) we compute

$$F''(0) = \operatorname{Trace} \tilde{f}''(a)u^2 .$$

Since $f'' \geq 0$, then $\tilde{f}''(a)$ is in $\bar{S}_+$, as well as $u\tilde{f}''(a)u$. Thus $F''(0) \geq 0$. This implies that (1.6) is convex. The case where f'' does not exist is then treated by approximation.

To come back to our initial problem, i.e. a suitable generalization of (1.1) through (1.2), we consider (1.6) when f is the logarithm of the Laplace transform L of some positive measure μ on $\mathbb{R}$. Let us assume that for all λ in the open interval I

$$L(\lambda) = \exp f(\lambda) = \int_{\mathbb{R}} \exp(-\lambda x)\, \mu(dx) < \infty \tag{1.7}$$

It is well known that f is convex on I. Thus, as we have seen, (1.6) is convex, and one can wonder if there exists a positive measure $\tilde{\mu}$ on S such that for all a in $S(I)$ one has

$$\operatorname{Det} \tilde{L}(a) = \exp \operatorname{Trace} \tilde{f}(a) = \int_S \exp\left(-\operatorname{Trace}(ax)\right) \tilde{\mu}(dx) \tag{1.8}$$

An instance for which it is true is the case $I = \mathbb{R}$ and $f(\lambda) = \sigma^2\lambda^2/2$: clearly $\tilde{\mu}$ is a suitable Gaussian distribution on S. An other instance for which it is almost true is the case where $I = (0, +\infty)$ and $f(\lambda) = -p \operatorname{Log} \lambda$, where $p > 0$. Here (1.7) holds with

$$\mu(dx) = \frac{x^{p-1}}{\Gamma(p)} \mathbb{1}_{(0,+\infty)}(x)\, dx .$$

However $\tilde{\mu}$ defined by (1.8) will be positive if and only if

$$p \in \left\{\tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots, \tfrac{n-1}{2}\right\} \cup \left(\tfrac{n-1}{2}, +\infty\right) \tag{1.9}$$

This result (1.9) is due to Gindikin (1975). It has been rediscovered again and again : see Casalis and Letac (1994) for references, and a short proof.

We are now able to state the aim of this note : to study the existence of a positive $\tilde{\mu}$ in (1.8) when $I = (0, +\infty)$ and $f(\lambda) = -p\sqrt{2\lambda}$ (compare (1.2) and (1.7)). As we shall see (section 5) the answer is negative for $n \geq 2$, and we shall prove this by computing explicitely a signed measure $\tilde{\mu}$ such that (1.8) holds when $n = 2$. Explicit calculations for $n \geq 3$ seem hopeless. Section 2 is devoted to a general study of (1.8). Section 3 specializes to $n = 2$. Section 4 studies an integral equation that we meet by considering the case $f(\lambda) = -p\sqrt{2\lambda}$ and a slight extension of the problem to the Lorentz cone, which appears in section 5.

§2. Properties of $\tilde{\mu}$ for general n. We keep the notations of the introduction; furthermore we denote by $\mathbb{O}(E)$ and $\mathbb{O}(S)$ the orthogonal groups of the Euclidean spaces E and S. There is a natural representation of $\mathbb{O}(E)$ in $\mathbb{O}(S)$ defined as follows : if u is in $\mathbb{O}(E)$, then for all a in S, $g_u(a) = uau^{-1}$ is in S.

Furthermore $\operatorname{Trace}\left(g_u(a)\right)^2 = \operatorname{Trace} a^2$, thus g_u is in $\mathbb{O}(S)$. An argument of convexity shows easily that if u is in the subgroup $\mathbb{O}_+(E)$ of rotations of $\mathbb{O}(E)$, then g_u is in $\mathbb{O}_+(S)$ too. Clearly $g_{u_1}g_u = g_{u_1u}$, and $u \mapsto g_u$ defines an homomorphism from $\mathbb{O}(E)$ to $\mathbb{O}(S)$ and from $\mathbb{O}_+(E)$ to $\mathbb{O}_+(S)$. Note also that

$$uau^{-1} = a \quad \text{for all } u \text{ in } \mathbb{O}_+(E) \qquad \Longleftrightarrow \qquad a \in \mathbb{R}.\mathrm{id}_E \tag{2.1}$$

$$uau^{-1} = a \quad \text{for all } a \text{ in } S \qquad \Longleftrightarrow \qquad u = \pm\,\mathrm{id}_E \tag{2.2}$$

Denote by G and G_+ the respective images of $\mathbb{O}(E)$ and $\mathbb{O}_+(E)$ in $\mathbb{O}(S)$ by $u \mapsto g_u$. It is easy to see that a and b in S are in the same G_+ orbit —thus in the same G orbit— if and only if their spectrum coincide. More precisely if $\lambda_1(a) \leq \lambda_2(a) \leq \ldots \leq \lambda_n(a)$ is the sequence of not necessarily distinct eigenvalues of a, then there exists u in $\mathbb{O}_+(E)$ such that $b = uau^{-1}$ if and only if $\lambda_j(a) = \lambda_j(b)$ $j = 1, \ldots, n$. The necessary condition is clear; to prove the sufficient condition, if e and f are orthonormal basis of E such $a(\vec{e}_j) = \lambda_j(a)\vec{e}_j$ and $b(\vec{f}_j) = \lambda_j(b)\vec{f}_j$ then one takes u in $\mathbb{O}_+(E)$ such that $u(\vec{f}_j) = \vec{e}_j$. However, if such a u has determinant -1, one has to replace $\vec{f}_1$ by $-\vec{f}_1$, still an eigenvector of b.

Assume now that I and μ are as in (1.7) and suppose that (1.8) holds with a signed measure $\tilde{\mu}$. For u in $\mathbb{O}(E)$ we have :

$$\operatorname{Trace} \tilde{f}(a) = \operatorname{Trace} \tilde{f}\left(g_u(a)\right) .$$

Thus (1.8) becomes

$$\begin{aligned} &\int_S \exp\left(-\operatorname{Trace}(ax)\right)\tilde{\mu}(dx) = \exp \operatorname{Trace} \tilde{f}(a) = \exp \operatorname{Trace} \tilde{f}\left(g_u(a)\right) \\ &= \int_S \exp\left(-\operatorname{Trace}\left(a g_{u^{-1}}(x)\right)\right)\tilde{\mu}(dx) = \int_S \exp\left(-\operatorname{Trace}(ay)\right)\tilde{\mu}_1(dy) \end{aligned}$$

where $\tilde{\mu}_1(dy)$ is the image of $\tilde{\mu}$ by $x \mapsto y = g_{u^{-1}}(x)$.

Thus $\tilde{\mu}$ is invariant by G and G_+. Now S is split by G_+ in orbits and the set of these orbits is parametrized by the increasing sequence of the eigenvalues of any element of the orbit, i.e. by

$$H = \left\{h \in \mathbb{R}^n \; ; \; h_1 \leq h_2 \leq \ldots \leq h_n\right\} .$$