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Maassen Kernels and Self-Similar Quantum Fields

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Abstract. — In his Lecture Notes [Maj] P. Major has outlined a theory of multiple Wiener-Itô integrals with respect to a stationary Gaussian random field ξ over the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ of rapidly decreasing smooth functions in $\mathbb{R}^d$. Furthermore, he has exploited the same to construct self-similar random fields subordinate to ξ. Here, we observe that the Hilbert space of functions square integrable with respect to the probability measure P of ξ can be identified in a natural way with the Hilbert space of functions square integrable with respect to the symmetric Guichardet measure [Gui] constructed from the spectrum of ξ. Under such an identification, multiplication of random variables on the probability space of ξ becomes the twisted convolution of Lindsay and Maassen [Li M 1,2] for Maassen kernels [Maa], [Mey]. The multiple Wiener-Itô integral of Major is described neatly by a twisted version of Meyer's multiplication formula (see (IV.4.1 in [Mey]). Following Lindsay and Parthasarathy [Li P] we introduce the weighted and twisted convolution of Maassen kernels, present a generalization of Meyer's formula and exploit it to construct a family of operator fields whose expectations in the vacuum state exhibit a simultaneous self-similarity property. Such a construction includes Major's examples and at the same time yields a self-similar Clifford field.

1 An involutive Gaussian random field and the Lindsay-Maassen twisted convolution algebra

Let $(X, \mathcal{F}, m)$ be a σ-finite measure space equipped with an m-preserving involution $x \to \tilde{x}$ on X satisfying $(\tilde{x})^{\sim} \equiv x$. For any measure μ, denote by $L^2_{\mathbb{R}}(\mu)$ and $L^2(\mu)$ respectively the real and complex Hilbert spaces of functions square integrable with respect to μ. Then the following holds:

Theorem 1.1 There exists a probability space $(\Omega, \mathcal{F}_m, P_m)$ and a linear map $\xi : L^2_{\mathbb{R}}(m) \to L^2(P_m)$ satisfying the following:

(a) For each $f \in L^2_{\mathbb{R}}(m)$, $\xi(f)$ is a complex-valued Gaussian random variable of mean 0.

(b) For any $f, g \in L^2_{\mathbb{R}}(m)$,

$$\mathbb{E}\,\overline{\xi(f)}\xi(g) = \int f(x)g(x)dm(x).$$

(c) If $\tilde{f}(x) \equiv f(\tilde{x})$ and $f \in L^2_{I\!R}(m)$ then $\xi(\tilde{f}) = \overline{\xi(f)}$.

(d) The σ algebra generated by $\{\xi(f), f \in L^2_{I\!R}(m)\}$ is $\mathcal{F}_m$.

Proof: For any $f, g \in L^2_{I\!R}(m)$ define

$$K_\pm(f,g) = \int \frac{1}{2}(f(x) \pm f(\tilde{x}))g(x)dm(x). \tag{1.1}$$

From the $\sim$ - invariance of m and Schwarz's inequality we have $K_\pm(f,g) = K_\pm(g,f)$,

$$\left|\int f(\tilde{x})f(x)dm(x)\right| \leq \int f^2(x)dm(x)$$

and therefore

$$K_\pm(f,f) = \frac{1}{2}\int (f^2(x) \pm f(\tilde{x})f(x))dm(x) \geq 0.$$

In other words K_+ and K_- are non-negative definite bilinear forms on $L^2_{I\!R}(m)$ with non-trivial kernel (consisting of odd functions for K_+ and even functions for K_-). Hence there exist two independent real Gaussian random fields ξ_+ and ξ_- over $L^2_{I\!R}(m)$ on some probability space $(\Omega, \mathcal{F}_m, P_m)$ for which

$$I\!E\xi_\pm(f) = 0, \;\; I\!E\xi_+(f)\xi_+(g) = K_+(f,g), I\!E\xi_-(f)\xi_-(g) = K_-(f,g) \tag{1.2}$$

and $\mathcal{F}_m$ is generated by $\{\xi_+(f), \xi_-(f), f \in L^2_{I\!R}(m)\}$. Elementary algebra using (1.1), (1.2) and $\sim$-invariance of m yields

$$I\!E(\xi_+(f) - \xi_+(\tilde{f}))^2 = I\!E(\xi_-(f) + \xi_-(\tilde{f}))^2 = 0 \tag{1.3}$$

where $\tilde{f}(x) = f(\tilde{x})$. Define

$$\xi(f) = \xi_+(f) + i\xi_-(f).$$

Clearly, ξ is a linear map satisfying (a) and (c). Furthermore

$$I\!E\,\overline{\xi(f)}\,\xi(g) = K_+(f,g) + K_-(f,g) = \int f(x)g(x)dm(x)$$

proving (b). Property (d) is immediate. ∎

Corollary 1.2 Let $\{\xi(f), f \in L^2_{I\!R}(m)\}$ be as in Theorem 1.1. For any f in the complex Hilbert space $L^2(m)$ with $f = f_1 + if_2$, where f_1 and f_2 are respectively the real and imaginary parts of f, let $\xi(f) = \xi(f_1) + i\xi(f_2)$. Then $\{\xi(f), f \in L^2(m)\}$ satisfies the following:

(a) The correspondence $f \to \xi(f)$ is complex linear.

(b) For each $f, \xi(f)$ is a complex-valued Gaussian random variable of mean 0.

(c) $\mathbb{E}\, \overline{\xi(f)}\, \xi(g) = \int \overline{f}(x) g(x) dm(x)$.

(d) If $\tilde{f}(x) \equiv \overline{f(\tilde{x})}$, then $\xi(\tilde{f}) = \overline{\xi(f)}$.

(e) $\mathbb{E} e^{\xi(f)} = \exp\, \frac{1}{2} \int f(\tilde{x}) f(x) dm(x)$.

Proof: The first four parts (a) - (d) are immediate from Theorem 1.1. The last part follows from the $\sim$-invariance of m and the relation

$$\xi(f) = \xi_+(f_1) + i\xi_+(f_2) + i(\xi_-(f_1) + i\xi_-(f_2))$$

where ξ_+ and ξ_- are the independent real Gaussian random fields over $L^2_{\mathbb{R}}(m)$ with respective covariance kernels K_+ and K_- in the proof of Theorem 1.1. ∎

Remark 1.3 In Corollary 1.2 define the normalised exponential random variable $e_\xi(f)$ by

$$e_\xi(f) = \exp(\xi(f) - \frac{1}{2} \int f(\tilde{x}) f(x) dm(x)) \tag{1.4}$$

for $f \in L^2(m)$. Then $\{e_\xi(f), f \in L^2(m)\}$ is a linearly independent and total set in $L^2(P_m)$. Furthermore

$$\mathbb{E}\, \overline{e_\xi(f)}\, e_\xi(g) = \exp \int \overline{f(x)}\, g(x) dm(x), \tag{1.5}$$

$$e_\xi(f) e_\xi(g) = e_\xi(f+g) \exp \int f(\tilde{x}) g(x) dm(x) \tag{1.6}$$

for all $f, g \in L^2(m)$.

We shall denote by $\mathcal{E}_\xi \subset L^2(P_m)$ the dense linear manifold generated by $\{e_\xi(f), f \in L^2(m)\}$. Then (1.6) implies that $\mathcal{E}_\xi$ is an algebra of random variables on $(\Omega, \mathcal{F}_m, P_m)$. Owing to property (d) in Corollary 1.2 we may call ξ an *involutive Gaussian random field.*

From now on we assume that $(X, \mathcal{F}, m)$ is a separable, nonatomic and σ-finite measure space. Our aim is to identify $L^2(P_m)$ in Theorem 1.1 with $L^2(m_\Gamma)$ where m_Γ is the symmetric measure of Guichardet [Gui] in the space $\Gamma(X)$ of all finite subsets of X, constructed from m. We denote the Guichardet symmetric measure space by $(\Gamma(X), \mathcal{F}_\Gamma, m_\Gamma)$ so that integration with respect to m_Γ is determined by

$$\int_{\Gamma(x)} f(\sigma) dm_\Gamma(\sigma) = f(\emptyset) + \sum_{n=1}^{\infty} \frac{1}{n!} \int f(\{x_1, x_2, \cdots, x_n\}) m(dx_1) \cdots m(dx_n) \tag{1.7}$$

for any $f \in L^1(m_\Gamma)$ where, on the right hand side, $f(\{x_1, x_2, \cdots, x_n\})$ is viewed as a symmetric measurable function of n variables $x_1, x_2, ..., x_n$ with all the $x_i's$ distinct. It is to be noted that the n-fold product of the nonatomic measure m has its support in the subset $\{(x_1, x_2, ..., x_n) : x_i \in X \text{ and } x_i \neq x_j \text{ if } i \neq j\}$. Denote by $\Gamma^n(X)$ the n-fold cartesian product of $\Gamma(X)$ and by $\Gamma^{(n)}(X) \subset \Gamma^n(X)$ the subset

$$\{\underline{\sigma} = (\sigma_1, \sigma_2, ..., \sigma_n) | \sigma_i \in \Gamma(X), \sigma_i \cap \sigma_j = \emptyset \text{ if } i \neq j\}.$$

Then the product measure m_Γ^n satisfies $m_\Gamma^n(\Gamma^n(X) \backslash \Gamma^{(n)}(X)) = 0$. For simplicity we write $d\sigma = dm_\Gamma(\sigma)$ in $\Gamma(X)$. If $\sigma_1, \sigma_2, ..., \sigma_n$ are disjoint elements of $\Gamma(X)$ we write $\sigma_1 + \sigma_2 + \cdots + \sigma_n$ or $\sum_{i=1}^n \sigma_i$ to denote $\bigcup_{i=1}^n \sigma_i$. Then one has the following Maassen's sum-integral formula for $f \in L^1(m_\Gamma^n)$:

$$\int_{\Gamma^{(n)}(X)} f(\sigma_1, \sigma_2, ..., \sigma_n) d\sigma_1 d\sigma_2 ... d\sigma_n = \int_{\Gamma(X)} \{ \sum_{\sigma_1 + \cdots + \sigma_n = \sigma} f(\sigma_1, ..., \sigma_n)\} d\sigma. \tag{1.8}$$

For a proof see [Mey], [Li P]. Following [Maa] we introduce the space $\mathcal{K}(X) = \mathcal{K}(X, m, \sim) \subset L^2(m_\Gamma)$ of Massen kernels:

$$\mathcal{K}(X) = \{f | \int a^{\#\sigma} |f(\sigma)|^2 d\sigma < \infty \ \forall \ a > 1\}. \tag{1.9}$$

The Lindsay-Maassen twisted convolution $f * g$ between any two Maassen kernels f and g is defined by

$$(f * g)(\sigma) = \sum_{\sigma_1 + \sigma_2 = \sigma} \int f(\sigma_1 + \tilde{\omega}) g(\omega + \sigma_2) d\omega \tag{1.10}$$

where the summation on the right hand side is over all partitions of σ into a pair σ_1, σ_2 of subsets (which can be empty). Then $f * g \in \mathcal{K}(X)$ and satisfies the inequality

$$\int |a^{\#\sigma}(f * g)(\sigma)|^2 d\sigma \leq \int |(a\sqrt{3})^{\#\sigma} f(\sigma)|^2 d\sigma \cdot \int |(a\sqrt{3})^{\#\sigma} g(\sigma)|^2 d\sigma \text{ for all } a \geq 1. \tag{1.11}$$

For a proof see Proposition 3.2 in [Li P]. The $\sim$ - invariance of m implies the invariance of the associated Guichardet measure m_Γ on $\Gamma(X)$ under the involution transformation $\omega \to \tilde{\omega} = \{\tilde{x} | x \in \omega\}$ and hence it is clear from (1.10) that $f * g = g * f$. It follows from the sum-integral formula (1.8) that $*$ is even associative. This will also follow from our Theorem 1.4. Thus $\mathcal{K}(X)$ becomes a commutative and associative algebra equipped with the involution $f \to \tilde{f}$ where $\tilde{f}(\sigma) = \overline{f(\tilde{\sigma})}$. A simple computation shows that $(f * g)^\sim = \tilde{f} * \tilde{g}$.

For any $\varphi \in L^2(m)$ define the associated exponential kernel $e(\varphi) \in \mathcal{K}(X)$ by

$$e(\varphi)(\sigma) = \begin{cases} 1 & \text{if } \sigma = \emptyset \text{ ,} \\ \prod_{x \in \sigma} \varphi(x) & \text{otherwise.} \end{cases} \tag{1.12}$$

Then

$$e(\varphi) * e(\psi) = e(\varphi + \psi) \exp \int \varphi(\tilde{x}) \psi(x) dm(x), \tag{1.13}$$