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TAMAGAWA NUMBERS OF POLARIZED ALGEBRAIC VARIETIES

by

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Dedicated to Professor Yu. I. Manin on his 60th birthday

Abstract. — Let $\mathcal{L} = (L, \|\cdot\|_v)$ be an ample metrized invertible sheaf on a smooth quasi-projective algebraic variety V over a number field F. Denote by $N(V, \mathcal{L}, B)$ the number of rational points in V having $\mathcal{L}$-height $\leqslant B$. In this paper we consider the problem of a geometric and arithmetic interpretation of the asymptotic for $N(V, \mathcal{L}, B)$ as $B \to \infty$ in connection with recent conjectures of Fujita concerning the Minimal Model Program for polarized algebraic varieties.

We introduce the notions of *$\mathcal{L}$-primitive varieties* and *$\mathcal{L}$-primitive fibrations*. For $\mathcal{L}$-primitive varieties V over F we propose a method to define an adelic Tamagawa number $\tau_{\mathcal{L}}(V)$ which is a generalization of the Tamagawa number $\tau(V)$ introduced by Peyre for smooth Fano varieties. Our method allows us to construct Tamagawa numbers for $\mathbf{Q}$-Fano varieties with at worst canonical singularities.

In a series of examples of smooth polarized varieties and singular Fano varieties we show that our Tamagawa numbers express the dependence of the asymptotic of $N(V, \mathcal{L}, B)$ on the choice of v-adic metrics on $\mathcal{L}$.

1. Introduction

Let F be a number field (a finite extension of $\mathbf{Q}$), $\mathrm{Val}(F)$ the set of all valuations of F, F_v the v-adic completion of F with respect to $v \in \mathrm{Val}(F)$, and $|\cdot|_v : F_v \to \mathbf{R}$ the v-adic norm on F_v normalized by the conditions $|x|_v = |N_{F_v/\mathbf{Q}_p}(x)|_p$ for p-adic valuations $v \in \mathrm{Val}(F)$.

Consider a projective space $\mathbf{P}^m$ with homogeneous coordinates $(z_0, ..., z_m)$ and a locally closed quasi-projective subvariety $V \subset \mathbf{P}^m$ defined over F (we want to stress that V is not assumed to be projective). Let $V(F)$ be the set of points in V

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with coordinates in F. A **standard height function** $H : \mathbf{P}^m(F) \to \mathbf{R}_{>0}$ is defined as follows

$$H(x) := \prod_{v \in \mathrm{Val}(F)} \max_{j=0,\ldots,m} \{|z_j(x)|_v\}.$$

A basic fact about the standard height function H claims that the set

$$\{x \in \mathbf{P}^m(F) \,:\, H(x) \leq B\}$$

is finite for any real number B [**29**]. We set

$$N(V, B) = \#\{x \in V(F) \,:\, H(x) \leq B\}.$$

It is an experimental fact that whenever one succeeds in proving an asymptotic formula for the function $N(V, B)$ as $B \to \infty$, one obtains the asymptotic

$$N(V, B) = c(V) B^{a(V)} (\log B)^{b(V)-1} (1 + o(1)) \tag{1}$$

with some constants $a(V) \in \mathbf{Q}$, $b(V) \in \frac{1}{2}\mathbf{Z}$, and $c(V) \in \mathbf{R}_{>0}$. We want to use this observation as our starting point. It seems natural to ask the following:

Question A. — *For which quasi-projective subvarieties $V \subset \mathbf{P}^m$ defined over F do there exist constants $a(V) \in \mathbf{Q}$, $b(V) \in \frac{1}{2}\mathbf{Z}$ and $c(V) \in \mathbf{R}_{>0}$ such that the asymptotic formula* (1) *holds?*

Question B. — *Does there exist a quasi-projective variety V over F with an asymptotic which is different from* (1)*?*

In this paper we will be interested not in Questions A and B themselves but in a related to them another natural question:

Question C. — *Assume that V is an irreducible quasi-projective variety over a number field F such that the asymptotic formula* (1) *holds. How to compute the constants $a(V), b(V)$ and $c(V)$ in this formula via some arithmetical properties of V over F and geometrical properties of V over $\mathbf{C}$?*

To simplify our terminology, it will be convenient for us to postulate:

Assumption. — For all quasi-projective V', V with $V' \subset V \subset \mathbf{P}^m$ and with $|V(F)| = \infty$ there exists the limit

$$\lim_{B \to \infty} \frac{N(V', B)}{N(V, B)}.$$

The following definitions have been useful to us:

Definition S_1. — A smooth irreducible quasi-projective subvariety $V \subset \mathbf{P}^m$ over a number field F is called **weakly saturated**, if $|V(F)| = \infty$ and if for any locally closed subvariety $W \subset V$ with $\dim W < \dim V$ one has

$$\lim_{B\to\infty} \frac{N(W,B)}{N(V,B)} < 1.$$

It is important to remark that Question C really makes sense *only for weakly saturated* varieties. Indeed, if there were a locally closed subvariety $W \subset V$ with $\dim W < \dim V$ and

$$\lim_{B\to\infty} \frac{N(W,B)}{N(V,B)} = 1,$$

then it would be enough to answer Question C for each irreducible component of W and for all possible intersections of these components (i.e., one could forget about the existence of V and reduce the situation to a lower-dimensional case). In general, it is not easy to decide whether or not a given locally closed subvariety $V \subset \mathbf{P}^m$ is weakly saturated. We expect (and our assumption implies this) that the orbits of connected subgroups $G \subset PGL(m+1)$ are examples of weakly saturated varieties $V \subset \mathbf{P}^m$ (see 3.2.8).

Definition S_2. — A smooth irreducible quasi-projective subvariety $V \subset \mathbf{P}^m$ with $|N(V,B)| = \infty$ is called **strongly saturated**, if for all dense Zariski open subsets $U \subset V$, one has

$$\lim_{B\to\infty} \frac{N(U,B)}{N(V,B)} = 1.$$

First of all, if $V \subset \mathbf{P}^m$ is a strongly saturated subvariety, then for any locally closed subvariety $W \subset V$ with $\dim W < \dim V$, one has

$$\lim_{B\to\infty} \frac{N(W,B)}{N(V,B)} = 0,$$

i.e., V is weakly saturated.

On the other hand, if $V \subset \mathbf{P}^m$ is weakly saturated, but not strongly saturated, then there must be an infinite sequence $W_1, W_2, \ldots$ of pairwise different locally closed irreducible subvarieties $W_i \subset V$ with $\dim W_i < \dim V$ and $|W_i(F)| = \infty$ such that for an arbitrary positive integer k one has

$$0 < \lim_{B\to\infty} \frac{N(W_1 \cup \cdots \cup W_k, B)}{N(V,B)} < 1.$$

Moreover, in this situation one can always choose the varieties W_i to be strongly saturated (otherwise one could find $W_i' \subset W_i$ with $\dim W_i' < \dim W_i$ with the same properties as W_i etc.). The strong saturatedness of each W_i implies that

$$\lim_{B\to\infty} \frac{N(W_{i_1} \cap \cdots \cap W_{i_l}, B)}{N(V,B)} = 0$$

for all pairwise different $i_1, \dots, i_l$ and $l \geqslant 2$. In particular, one has

$$\sum_{i=1}^{k} \lim_{B\to\infty} \frac{N(W_i, B)}{N(V, B)} = \lim_{B\to\infty} \frac{N(W_1 \cup \dots \cup W_k, B)}{N(V, B)} < 1 \ \ \forall k > 0.$$

Definition F. — Let V be a weakly saturated quasi-projective variety in $\mathbf{P}^m$ and $W_1, W_2, \dots$ an infinite sequence of strongly saturated irreducible subvarieties W_i having the property

$$0 < \theta_i := \lim_{B\to\infty} \frac{N(W_i, B)}{N(V, B)} < 1 \ \ \forall i > 0.$$

We say that the set $\{W_1, W_2, \dots\}$ forms an **asymptotic arithmetic fibration** on V, if the following equality holds

$$\sum_{i=1}^{\infty} \theta_i = 1.$$

The main purpose of this paper is to explain some geometric and arithmetic ideas concerning weakly saturated varieties and their asymptotic arithmetic fibrations by strongly saturated subvarieties. It seems that the cubic bundles considered in [**9**] are examples of such a fibration. We want to remark that most of the above terminology grew out of our attempts to restore a conjectural picture of the interplay between the geometry of algebraic varieties and the arithmetic of the distribution of rational points on them after we have found in [**9**] an example which contradicted general expectations formulated in [**4**].

In section 2 we consider smooth quasi-projective varieties V over $\mathbf{C}$ together with a polarization $\mathcal{L} = (L, \|\cdot\|_h)$ consisting of an ample line bundle L on V equipped with a positive hermitian metric $\|\cdot\|_h$. Our main interest in this section is a discussion of geometric properties of V in connection with the Minimal Model Program [**27**] and its version for polarized algebraic varieties suggested by Fujita [**20, 23, 24**]. We introduce our main geometric invariants $\alpha_{\mathcal{L}}(V)$, $\beta_{\mathcal{L}}(V)$, and $\delta_{\mathcal{L}}(V)$ for an arbitrary $\mathcal{L}$-polarized variety V. It is important to remark that we will be only interested in the case $\alpha_{\mathcal{L}}(V) > 0$. The number $\alpha_{\mathcal{L}}(V)$ was first introduced in [**4, 5**], it equals to the opposite of the so called *Kodaira energy* (investigated by Fujita in [**20, 23, 24**]). Our basic geometric notion in the study of $\mathcal{L}$-polarized varieties V with $\alpha_{\mathcal{L}}(V) > 0$ is the notion of an *$\mathcal{L}$-primitive* variety. In Fujita's program for polarized varieties with negative Kodaira energy $\mathcal{L}$-primitive varieties play the same role as $\mathbf{Q}$-Fano varieties in Mori's program for algebraic varieties with negative Kodaira dimension. In particular, one expects the existence of so called *$\mathcal{L}$-primitive fibrations*, which are analogous to $\mathbf{Q}$-Fano fibrations in Mori's program. We show that on $\mathcal{L}$-primitive varieties there exists a canonical volume measure. Moreover, this measure allows us to construct a descent of hermitian metrics to the base of $\mathcal{L}$-primitive fibrations. Many geometric ideas of this section are inspired by [**4, 5**].