JEAN-MARC DESHOUILLERS
GREGORY A. FREIMAN

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ON AN ADDITIVE PROBLEM OF ERDŐS AND STRAUS, 2

by

Jean-Marc Deshouillers & Gregory A. Freiman

Abstract. — We denote by $s^\wedge A$ the set of integers which can be written as a sum of s pairwise distinct elements from A. The set A is called admissible if and only if $s \neq t$ implies that $s^\wedge A$ and $t^\wedge A$ have no element in common.

P. Erdős conjectured that an admissible set included in $[1, N]$ has a maximal cardinality when A consists of consecutive integers located at the upper end of the interval $[1, N]$. The object of this paper is to give a proof of Erdős' conjecture, for sufficiently large N.

Let $\mathcal{A}$ be a set of positive integers having the property that each time an integer n can be written as a sum of distinct elements of $\mathcal{A}$, the number of summands is well defined, in that the integer n cannot be written as a sum of distinct elements of $\mathcal{A}$ with a different number of summands. This notion has been introduced by P. Erdős in 1962 (cf. [**2**]) and called **admissibility** by E.G. Straus in 1966 (cf. [**5**]). In other words, if we denote by $s^\wedge\mathcal{A}$ the set of integers which can be written as a sum of s pairwise distinct elements from $\mathcal{A}$ then $\mathcal{A}$ is **admissible** if and only if $s \neq t$ implies that $s^\wedge\mathcal{A}$ and $t^\wedge\mathcal{A}$ have no element in common.

Erdős conjectured that an admissible subset $\mathcal{A}$ included in $[1, N]$ has a cardinality which is maximal when $\mathcal{A}$ consists of consecutive integers located at the upper end of the interval $[1, N]$. As it was computed by E.G. Straus, the set

$$\{N - k + 1, N - k + 2, \dots, N\}$$

is admissible if and only if $k \leq 2\sqrt{N + 1/4} - 1$.

Straus himself proved that $\sqrt{N}$ is the right order of magnitude for the cardinality of a maximal admissible subset from $[1, N]$. More precisely, he proved the inequality $|\mathcal{A}| \leq (4/\sqrt{3} + o(1))\sqrt{N}$. The constant involved has been slightly reduced by P. Erdős, J-L. Nicolas and A. Sárkőzy (cf. [**3**]) and we proved (cf. [**1**]) the inequality

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$|\mathcal{A}| \leq (2 + o(1))\sqrt{N}$. The object of this paper is to give a proof of Erdős conjecture, at least when N is sufficiently large.

Theorem 1. — *There exists an integer N_0, effectively computable, such that for any integer $N \geq N_0$ and any admissible subset $\mathcal{A} \subset [1, N]$ we have*

$$\operatorname{Card} \mathcal{A} \leq 2\sqrt{N + 1/4} - 1.$$

The proof is based on the description of the structure of large admissible sets we obtained previously, namely :

***Theorem 2* (J-M. Deshouillers, G.A. Freiman [1]).** — *Let $\mathcal{A}$ be an admissible set included in $[1, N]$, such that* $\operatorname{Card} \mathcal{A} > 1.96\sqrt{N}$. *If N is large enough, there exist $\mathcal{C} \subset \mathcal{A}$ and an integer q having the following properties :*
(i) $\operatorname{Card} \mathcal{C} \leq 10^5 N^{5/12}$,
(ii) for some t the set $t^\wedge\mathcal{C}$ contains at least $3N^{5/6}$ terms in an arithmetic progression modulo q,
(iii) $\mathcal{A} \setminus \mathcal{C}$ is included in an arithmetic progression modulo q containing at most $N^{7/12}$ terms.

Although we do not develop this point, it will be clear from the proof that our arguments may be used to describe the structure of maximal admissible subsets of $[1, N]$, leading for example to the fact that when N has the shape n^2 or $n^2 + n$ (and n sufficiently large), the Erdős - Straus example is the only maximal subset of $[1, N]$.

1. We first establish a lemma expressing the fact that if a set of integers $\mathcal{D}$ is part of a finite arithmetic progression with few missing elements, then the same is locally true for $s^\wedge\mathcal{D}$.

Proposition 1. — *Let us consider integers r, s, t and a, q such that $t \geq 2s - q$, $s \geq 4r + 3 + q$ and $0 \leq a < q$.*
Let further $\mathcal{D} = \{d_1 < d_2 < \cdots < d_t\}$ be a set of t distinct integers congruent to a modulo q such that $d_t - d_1 = (t - 1 + r)q$,and denote by m (resp. M) the smallest (resp. largest) element in $s^\wedge\mathcal{D}$. Then, among $2r + 1$ consecutive integers congruent to sa modulo q and laying in the interval $[m, M]$, at least $r + 1$ belong to $s^\wedge\mathcal{D}$.

Proof. — We treat the special case when $a = 0, q = 1$ and $\mathcal{D}$ is included in $[1, t]$. We notice that the general case reduces to this one by writing
$d_l = d_1 + q(\delta_l - 1)$ and considering the set $\{\delta_1, \ldots, \delta_t\}$.

Let x be an integer in $s^\wedge\mathcal{D} \cap [m, (m + M)/2]$. We first show that the interval $[x, x + 3r]$ contains at least $2r + 1$ elements from $s^\wedge\mathcal{D}$. Since x is in $s^\wedge\mathcal{D}$, we can find $d(1) < \cdots < d(s)$, elements in $\mathcal{D}$, the sum of which is x.

Let us show that $d(1)$ is less than $t - s - 3r$. On the one hand we have

$$m + M \leq (r + 1) + \cdots + (r + s) + (t + r - s + 1) + \cdots + (t + r) = \frac{s}{2}(2t + 4r + 2),$$

and on the other hand we have

$$x \geq d(1) + (d(1) + 1) + \cdots + (d(1) + s - 1) = \frac{s}{2}(2d(1) + s - 1).$$

The inequality $x \leq (m+M)/2$ implies that we have

$$2d(1) + s - 1 \leq t + 2r + 1,$$

whence

$$2d(1) \leq 2(t - s - 3r) - (t - s - 4r - 2),$$

and we notice that $t - s - 4r - 2$ is positive, by the assumptions of Proposition 1.

Since $d(1)$ is less than $t - s - 3r$, the interval $[d(1), t + r]$ contains at least $s + 4r + 1$ integers. We denote by $i_1 < \cdots < i_l$ the indexes of those $d's$ such that $d(i_k + 1) - d(i_k) \geq 2$, with the convention that $d(i_l + 1) = 3Dt + r + 1$ in the case when $d(s) < t + r$. The set

$$\bigcup_{k=1}^{l}]d(i_k) + 1, d(i_k + 1) - 1[$$

contains at least $4r + 1$ integers. We now suppress from those intervals those which contain no element from $\mathcal{D}$, and we rewrite the remaining ones as

$$]d(j_1) + 1, d(j_1 + 1) - 1[, \ldots,]d(j_h) + 1, d(j_h + 1) - 1[.$$

They contain at least $3r + 1$ integers, among which at most r are not in $\mathcal{D}$.

Let us define u_1 to be the largest integer such that $d(j_1) + u_1$ is in $\mathcal{D}$ and is less than $d(j_1 + 1)$, and let us define $u_2, \ldots, u_h$ in a similar way. We consider the integers

$$x = y + d(j_1) + \cdots + d(j_h) \quad \text{(which defines } y\text{)},$$

$$x + 1 = y + d(j_1) + 1 + d(j_2) + \cdots + d(j_h),$$

$$\ldots$$

$$x + u_1 = y + d(j_1) + u_1 + d(j_2) + \cdots + d(j_h),$$

$$\ldots$$

$$x + u_1 + \cdots + u_h = y + d(j_1) + u_1 + d(j_2) + u_2 + \cdots + d(j_h) + u_h.$$

One readily deduces from this construction that the interval

$$[x, x + \min(3r, u_1 + \cdots + u_h)]$$

contains at most r elements which are not in $s^\wedge\mathcal{D}$.

What we have proven so far easily implies that any interval $[z - r, z]$ with $m \leq z \leq (M + m)/2$ contains at least one element in $s^\wedge\mathcal{D}$. Let us consider an interval $[y, y + 2r]$ with $m \leq y \leq (M + m)/2$. By what we have just said, the interval $[y - r, y]$ contains an element in $s^\wedge\mathcal{D}$, let us call it x. As we have shown the interval $[x, x + 3r]$ contains at most r integers not in $s^\wedge\mathcal{D}$, so that $[y, y + 2r]$ contains at most r integers not in $s^\wedge\mathcal{D}$, which is equivalent to say that it contains at least $r + 1$ elements from $s^\wedge\mathcal{D}$.

A similar argument taking into account decreasing sequences and starting with M shows that any interval $[y - 2r, y]$ with $(m + M)/2 \leq y \leq M$ contains at least $r + 1$ elements from $s^\wedge\mathcal{D}$.

2. We now prove the following result concerning the structure of a large admissible finite set.

Theorem 3. — *Let $\mathcal{A} = \{a_1 < \cdots < a_A\}$ be an admissible subset of $[1, N]$ with cardinality $A = 2N^{1/2} + O(N^{5/12})$, and let us define q to be the largest integer such that $\mathcal{A}$ is contained in an arithmetic progression modulo q. We have $q = O(N^{5/12})$ and there exists an integer u in $[N^{11/24}, 2N^{11/24}]$ such that*

$$a_{A-u} - a_{u+1} = q(2N^{1/2} + O(N^{11/24})).$$

Proof. — The proof is based on the structure result we quoted in the introduction as Theorem 2. We keep its notation and first show that an integer q satisfying (ii) and (iii) is indeed the largest integer such that $\mathcal{A}$ is contained in an arithmetic progression modulo q. We let $\mathcal{B}$ denote $\mathcal{A} \setminus \mathcal{C}$.

A simple counting argument will show that $\mathcal{A}$ is included in the same arithmetic progression as $\mathcal{B}$. Otherwise, let us consider an element $a \in \mathcal{A}$ which is not in the same arithmetic progression as $\mathcal{B}$ modulo q. The set $s^\wedge\mathcal{A}$ contains the disjoint sets $s^\wedge\mathcal{B}$ and $a + (s-1)^\wedge\mathcal{B}$. We thus have $|s^\wedge\mathcal{A}| \geq |s^\wedge\mathcal{B}| + |(s-1)^\wedge\mathcal{B}|$. It is well-known (cf. [4] for example) that $|s^\wedge\mathcal{B}| \geq s(|\mathcal{B}| - s)$ for $s \leq |\mathcal{B}|$, and since $\mathcal{A} \subset [1, N]$ is admissible we have

$$\begin{aligned} N(|\mathcal{B}|+1) &\geq \text{Card } \left(\textstyle\bigcup_s (s^\wedge\mathcal{B} \cup (a + (s-1)^\wedge\mathcal{B})\right) \\ &\geq 2\textstyle\sum_s |s^\wedge\mathcal{B}| \geq 2\sum_s s = 20(|\mathcal{B}| - s) = \frac{1}{3}|\mathcal{B}|^3 + O(N), \end{aligned}$$

which implies $|\mathcal{B}| \leq (\sqrt{3} + o(1))\sqrt{N}$, so that we have $|\mathcal{A}| = |\mathcal{B}| + |\mathcal{C}| \leq (\sqrt{3} + o(1))\sqrt{N}$, a contradiction.

We have so far proven that q divides $g := gcd(a_2 - a_1, \ldots, a_A - a_1)$. Property (ii) implies that q is a multiple of g, so that we have $q = g$, as we wished to show.

The second step in the proof consists in showing that for $0 < k \leq |\mathcal{B}| - q$, any element in $k^\wedge\mathcal{B}$ is less than any element in $(k+q)^\wedge\mathcal{B}$. Let us call J the $3N^{5/6}$ consecutive terms of the arithmetic progression modulo q, the existence of which is asserted in (ii). Since $\mathcal{B}$ is included in an arithmetic progression modulo q with less that $3N^{5/6}$ terms, the sets $k^\wedge\mathcal{B} + J$ and $(k+q)^\wedge\mathcal{B} + J$ consists of consecutive terms of arithmetic progressions modulo q, and moreover, they are in the *same* class modulo q. Since $\mathcal{A}$ is admissible, the sets $k^\wedge\mathcal{B} + J$ (included in $(k+t)^\wedge\mathcal{A}$) and $(k+q)^\wedge\mathcal{B} + J$ (included in $(k+q+t)^\wedge\mathcal{A}$) do not intersect. To prove that any element of $k^\wedge\mathcal{B}$ is less that any element of $(k+q)^\wedge\mathcal{B}$, it is now sufficient to notice that $k^\wedge\mathcal{B}$ contains an element (we can consider the smallest element of $k^\wedge\mathcal{B}$), which is smaller than some element of $(k+q)^\wedge\mathcal{B}$.

We now prove that $q = O(N^{5/12})$. The cardinality of $\mathcal{A}$ and Theorem 2 imply that $|\mathcal{B}| = 2N^{1/2} + O(N^{5/12})$. We choose k so that $2k + q$ is $|\mathcal{B}|$ or $|\mathcal{B}| - 1$. (We notice that this is always possible since $\mathcal{A}$ contains at least $N^{1/2}$ integers from $[1, N]$ in an arithmetic progression modulo q, so that $q \leq N^{1/2}$). By the second step, the largest element in $k^\wedge\mathcal{B}$ is smaller than the largest element in $(k+q)^\wedge\mathcal{B}$. Let z be $(k+q)$-th element from $\mathcal{B}$, in the increasing order. We have

$$z \leq N - (k-1)q$$