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STRATIFIED SETS

by

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Abstract. — A set $\mathcal{A}$ of integers is said "stratified" if, for all t, $0 \leq t < \operatorname{Card} \mathcal{A}$, the sum of any t distinct elements of $\mathcal{A}$ is smaller than the sum of any $t+1$ distinct elements of $\mathcal{A}$. That implies that all elements of $\mathcal{A}$ should be positive. It is proved that the number of stratified sets with maximal element equal to N is exactly the number $p(N)$ of partitions of N.

1. Introduction

Let $\mathbb{N} = \{1, 2, 3, \dots\}$ denote the set of positive integers. After Erdős and Straus (see **[3]** and **[7]**), a set $\mathcal{A} \subset \mathbb{N}$ is said admissible if for any pairs $\mathcal{A}_1, \mathcal{A}_2$ of subsets of $\mathcal{A}$, one has

$$\left(\sum_{a \in \mathcal{A}_1} a = \sum_{a \in \mathcal{A}_2} a \right) \Rightarrow \mid \mathcal{A}_1 \mid = \mid \mathcal{A}_2 \mid .$$

Here $\mid \mathcal{A} \mid$ will denote the number of elements of $\mathcal{A}$.

Straus has observed that, if $k = \lfloor 2\sqrt{N + 1/4} - 1 \rfloor$, then the set $\mathcal{A} = \{N, N-1, \dots, N-k+1\}$ is admissible, On the other hand, he proved (cf. **[7]**) that if $N = \max_{a \in \mathcal{A}} a$, and $\mathcal{A}$ is admissible, then $\mid \mathcal{A} \mid \leq \left(\frac{4}{\sqrt{3}} + o(1) \right) \sqrt{N}$. The constant $4/\sqrt{3}$ has been improved in **[4]**, and in **[1]**, J.M. Deshouillers and G.A. Freiman have replaced it by the best possible constant 2. In **[2]**, they prove that for N large enough, the above example of Straus is the greatest possible admissible set with maximal element N.

Definition 1. — *A set $\mathcal{A} \subset \mathbb{Z}$ is stratified, if for $0 \leq t < t'$ the sum of any t distinct elements of $\mathcal{A}$ is strictly smaller than the sum of any t' distinct elements of $\mathcal{A}$.*

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Note that from the above definition all the elements of $\mathcal{A}$ should be positive (choose $t = 0$ and $t' = 1$), and so $\mathcal{A}$ is included in $\mathbb{N}$.

Clearly a stratified set is admissible. The above example of Straus is stratified, and in the table at the end of **[4]**, it can be seen that most of large admissible sets are stratified.

In this paper, stratified sets will be described in terms of partitions (Theorem 1). Further, we shall reformulate some of the conjectures about admissible sets given in **[4]** in terms of stratified sets. Finally, we shall show that the number of stratified sets with maximal element N is equal to the number of partitions of N (Theorem 2) and a one to one correspondence associating such a stratified set to a partition of N is explicited. As a corollary, the lower bound given in **[4]** for the total number of admissible sets with maximal element N will be improved.

It is possible to extend the notion of stratified set to subsets in arithmetic progression and in this way to describe some other classes of admissible sets. For instance a subset of odd numbers $\mathcal{A}$ which satisfies that the sum of any t distinct elements is smaller than the sum of any $t + 2$ distinct elements will certainly be admissible, since the sum of t elements and the sum of $(t+1)$ elements are of different parity and therefore are unequal. I hope to return to this question in an other paper.

At the end of this article, a table of the numbers $p(N)$ of stratified sets and $a(N)$ of admissible sets with largest element N is given. The table of $a(N)$ given in **[4]** is erroneous.

This work has started in September 1991, when G. Freiman was visiting me in Lyon. At that time he was trying to understand the structure of a large admissible set (it was before getting the result of **[1]**), and he wrote on the blackboard many equations like (11) or (12) below. So an important part of this paper is due to G. Freiman, and I thank him strongly.

I thank also very much Marc Deléglise for calculating the values of $a(N)$, and for listing stratified sets which drove me to Theorem 2. I thank also Paul Erdős, András Sárközy, Etienne Fouvry and Jean-Marc Deshouillers for fruitful discussions on this subject.

Notation. — $t^\wedge\mathcal{A}$ will denote the set of the sums of t distinct elements of $\mathcal{A}$.

2. Description of a stratified set

First it will be proved:

Proposition 1. — *Let $\mathcal{A} = \{a_1 < a_2 < \cdots < a_k = N\}$ be a set of positive integers, and $t_0 = \lfloor (k-1)/2 \rfloor$. Then $\mathcal{A}$ is stratified if and only if*

$$\max t_0{}^\wedge\mathcal{A} < \min(t_0+1)^\wedge\mathcal{A}. \tag{1}$$

Proof. — From the definition, $\mathcal{A}$ is stratified if for all $t, 1 \le t \le k-1$,

$$\max t^\wedge\mathcal{A} < \min(t+1)^\wedge\mathcal{A}. \tag{2}$$

Let us first prove (2) for $t \le t_0$. From (1), one has:

$$a_k + a_{k-1} + \cdots + a_{k-t_0+1} < a_1 + a_2 + \cdots + a_{t_0+1}$$

which implies

$$a_k + a_{k-1} + \cdots + a_{k-t+1} < a_1 + \cdots + a_{t+1} + \sum_{i=1}^{t_0-t} (a_{t+1+i} - a_{k-t+1-i}) \,. \tag{3}$$

But for $1 \le i \le t_0 - t$, we have $t+1+i < k-t+1-i$ since $2i \le 2(t_0 - t) \le k - 1 - 2t < k - 2t$; thus, the last sum in (3) is non-positive and (3) yields (2). Let us now suppose that $t > t_0$ and set $S = \sum_{i=1}^{k} a_i$ and $t' = k - t - 1$. We have $k/2 - 1 \le t_0 \le (k-1)/2$, so that

$$t' = k - t - 1 < k - t_0 - 1 \le k - (k/2 - 1) - 1 = k/2 \le t_0 + 1,$$

and so, $t' \le t_0$. From the above proof, one gets

$$\max(t')^\wedge \mathcal{A} < \min(t'+1)^\wedge \mathcal{A}, \tag{4}$$

and from the definition of t',

$$\max t^\wedge \mathcal{A} = S - \min(t'+1)^\wedge \mathcal{A} \tag{5}$$

and

$$\min(t+1)^\wedge \mathcal{A} = S - \max(t')^\wedge \mathcal{A}. \tag{6}$$

(4), (5) and (6) prove (2), and this completes the proof of Proposition 1.

Theorem 1

(a) Let k be even. There is a one to one correspondence between the stratified sets $\mathcal{A} \subset \mathbb{Z}$ with $\max \mathcal{A} = N$ and $|\mathcal{A}| = k$ and the solutions of the inequality

$$x_1 + 2(x_2 + x_{k-1}) + 3(x_3 + x_{k-2}) + \cdots + \frac{k}{2}(x_{k/2} + x_{k/2+1}) \le N - \frac{k^2}{4} - \frac{k}{2}. \tag{7}$$

where the x_i's are non negative integers.

(b) Let k be odd. There is a one to one correspondence between the stratified sets $\mathcal{A} \subset \mathbb{Z}$ with $\max \mathcal{A} = N$ and $|\mathcal{A}| = k$ and the solutions of the inequality:

$$x_1 + 2(x_2 + x_{k-1}) + \cdots + \frac{k-1}{2}(x_{(k-1)/2} + x_{(k+3)/2}) + \frac{k+1}{2}\, x_{(k+1)/2} \le N - \frac{(k+1)^2}{4}. \tag{8}$$

Proof. — Let $\mathcal{A} = \{a_1, a_2, \ldots, a_k\} \subset \mathbb{Z}$ with

$$a_1 < a_2 < \cdots < a_{k-1} < a_k = N. \tag{9}$$

be a stratified set. Let us introduce the new variables

$$x_i = a_{i+1} - a_i - 1, \quad 1 \le i \le k-1.$$

From (9), one has

$$x_i \geq 0, \quad 1 \leq i \leq k-1. \tag{10}$$

Conversely, (9) is clearly equivalent to $a_k = N$, and (10). Now we have to express (1) in terms of the $x_i's$, and this explains the role played by the parity of k.

Let us suppose k is even. From the definition of x_i, one has

$$x_1 + 2x_2 + \cdots + tx_t = -\frac{t(t+1)}{2} - a_1 - a_2 - \cdots - a_t + ta_{t+1} \tag{11}$$

$$\begin{aligned} 2x_{k-1} + 3x_{k-2} + \cdots + (u+1)x_{k-u} &= N + a_k + a_{k-1} + \cdots + a_{k-u+1} \\ &\quad -(u+1)a_{k-u} \\ &\quad -\frac{(u+1)(u+2)}{2} + 1. \end{aligned} \tag{12}$$

One chooses $t = t_0 + 1 = k/2$ in (11) and $u = t_0 = \frac{k}{2} - 1$ in (12) and then (11) and (12) give

$$\begin{aligned} \max t_0{}^\wedge\mathcal{A} - \min(t_0+1)^\wedge\mathcal{A} &= a_k + a_{k-1} + \cdots + a_{k-t_0+1} \\ &\quad -a_1 - a_2 - \cdots - a_{t_0+1} \\ &= x_1 + 2(x_2 + x_{k-1}) + \cdots \\ &\quad + \frac{k}{2}(x_{k/2} + x_{k/2+1}) \\ &\quad -N + \frac{k^2}{4} + \frac{k}{2} - 1. \end{aligned}$$

The last term -1 allows us to transform the strict inequality (2) in inequality (7) with $\leq$ sign.

The proof of (8) when k is odd is similar.

Corollary 1. — *Let us denote the number of stratified sets with k elements, and maximal element N by $S_k(N)$. The generating functions are:*
for k even

$$\sum_{N=0}^{\infty} S_k(N)x^N = x^{k^2/4+k/2} \prod_{i=1}^{k/2} \frac{1}{(1-x^i)^2}, \tag{13}$$

for k odd:

$$\sum_{N=0}^{\infty} S_k(N)x^N = \frac{x^{(k+1)^2/4}}{1-x^{(k+1)/2}} \prod_{i=1}^{(k-1)/2} \frac{1}{(1-x^i)^2}. \tag{14}$$

Proof. — It follows easily from the theorem, by the classical method of generating series. For k even, the generating series of the number of solutions of

$$x_1 + 2(x_2 + x_{k-1}) + \cdots + \frac{k}{2}(x_{k/2} + x_{(k/2+1)}) = n \tag{15}$$