Astérisque

YONUTZ V. STANCHESCU

On the structure of sets of lattice points in the plane with a small doubling property

Astérisque, tome 258 (1999), p. 217-240

<http://www.numdam.org/item?id=AST_1999__258__217_0>

Astérisque **258**, 1999, p. 217–240

ON THE STRUCTURE OF SETS OF LATTICE POINTS IN THE PLANE WITH A SMALL DOUBLING PROPERTY

by

Yonutz V. Stanchescu

Abstract. — We describe the structure of sets of lattice points in the plane, having a small doubling property. Let $\mathbb{K}$ be a finite subset of $\mathbb{Z}^2$ such that

$$|\mathbb{K}+\mathbb{K}| < 3.5|\mathbb{K}| - 7.$$

If $\mathbb{K}$ lies on three parallel lines, then the convex hull of $\mathbb{K}$ is contained in three compatible arithmetic progressions with the same common difference, having together no more than

$$|\mathbb{K}| + \frac{3}{4}\Big(|\mathbb{K}+\mathbb{K}| - \frac{10}{3}|\mathbb{K}| + 5\Big)$$

terms. This upper bound is best possible.

Notation

We write $[m,n] = \{x \in \mathbb{Z} |\ m \le x \le n\}$. For any nonempty finite set $K \subseteq \mathbb{R}$, $K = \{u_1 < u_2 < \cdots < u_k\}$ we denote by $k = |K|$ the *cardinality* of K and by $\ell(K)$ the *length* of K, that is the difference between its maximal and minimal elements. If $K \subseteq \mathbb{Z}$ and $k \ge 2$, by $d(K)$ we denote the *greatest common divisor* of $u_i - u_1$, $1 \le i \le k$. If $k = 1$, we put $d(K) = 0$. Let $h(K) = \ell(K) - |K| + 1$ denote the number of *holes* in K, that is $h(K) = |\ [u_1, u_k]\ \setminus\ K|$.

Let A and B be two subsets of an abelian group $(G,+)$. As usual, their *sum* is defined by $A + B = \{x \in G \mid x = a + b,\ a \in A,\ b \in B\}$ and we put $2A = A + A$. The *convex hull* of a set $\mathbb{S} \subseteq \mathbb{R}^2$ is denoted by $\mathrm{conv}(\mathbb{S})$. Vectors will be written in the form $u = (u_1, u_2)$, where u_1 and u_2 are the coordinates with respect to the canonical basis $e_1 = (1,0)$, $e_2 = (0,1)$.

1991 ***Mathematics Subject Classification.*** — 05D05, 11B75, 11P99.
Key words and phrases. — Two-dimensional lattice points, small doubling property.

1. Introduction

In additive number theory we usually ask what may be said about $M+M$, for a given set M. As a counterbalance to this direct approach, consider now the inverse problem: we study the properties of M, when some characteristic of $M+M$ is given, for example, the cardinality of the sum set $M+M$. It was noticed by Freiman [**F1**] that the assumption that $|2M|$ is small compared to $|M|$, implies strong restrictions on the structure of the set M. If $|2M| = 2|M| - 1$ and $M \subseteq \mathbb{Z}$, then M is an arithmetic progression. If we choose bigger values for $|2M|$, the problem ceases to be trivial. The fundamental theorem of G.A. Freiman [**F2**] gives the structure of finite sets of integers with small doubling property: $|2M| < c_0|M|$, where c_0 is any given positive number. This theorem was proved using geometric methods of number theory and a modification of the method of trigonometric sums. Y. Bilu recently studied in [**B**] a case when c_0 is a slowly growing function of $|M|$. The generalization to the case of different summands $M+N$, with a new proof, is to be found in the paper of I.Z. Ruzsa [**R**].

However, in the case of small values of the constant c_0, elementary methods yield sharper results. Let $\mathbb{K} \subseteq \mathbb{Z}^2$ be a finite set of lattice points. Two cases have been studied by G. A. Freiman [**F1**], pp.11, 28.

Theorem A. — *If* $|\mathbb{K}+\mathbb{K}| < 3|\mathbb{K}| - 3$, *then*
(1) $\mathbb{K}$ *lies on a straight line.*
(2) $\mathbb{K}$ *is contained in an arithmetic progression of no more than* $v = |\mathbb{K}+\mathbb{K}| - |\mathbb{K}| + 1$ *terms.* □

Theorem B. — *If* $|\mathbb{K}+\mathbb{K}| < \frac{10}{3}|\mathbb{K}| - 5$, $|\mathbb{K}| \geq 11$ *and* $\mathbb{K}$ *is not contained in a line, then*
(1) $\mathbb{K}$ *lies on two parallel straight lines.*
(2) $\mathbb{K}$ *is contained in two arithmetic progressions with the same common difference having together no more than* $v = |\mathbb{K}+\mathbb{K}| - 2|\mathbb{K}| + 3$ *terms.* □

The generalization of Theorems A(1) and B(1), to s lines, $s \geq 3$, was obtained in [**S2**]:

Theorem C. — *If* $|\mathbb{K}+\mathbb{K}| < \left(4 - \frac{2}{s+1}\right)|\mathbb{K}| - (2s+1)$ *and* $|\mathbb{K}| \geq 16s(s+1)(2s+1)$, *then there exist* s *parallel lines which cover the set* $\mathbb{K}$. □

A result which generalizes Theorems A(2) and B(2) was obtained in [**S3**].

Theorems A(1), B(1) and C cannot be sharpened by increasing the upper bound for $|2\mathbb{K}|$. (see Example A in [**S2**].) Assertion (2) of Theorems A and B gives the precise structure theorem for $s = 1$ and $s = 2$. In [**S2**] we obtained a sharpening of Theorem B(2) by giving the best possible value of the upper bound for $|2\mathbb{K}|$, under the additional assumption that $\mathbb{K}$ lies on $s = 2$ parallel lines. We proved that Theorem B(2) is true, even we replace $|2\mathbb{K}| < \frac{10}{3}|\mathbb{K}| - 5$ by $|2\mathbb{K}| < 4|\mathbb{K}| - 6$. More precisely:

Theorem S. — *Let* $\mathbb{K} \subseteq \mathbb{Z}^2$ *be a finite set, which lies on the lines* $x_2 = 0$ *and* $x_2 = 1$. *Let the set of abscissae for* $x_2 = 0$ *and* $x_2 = 1$, *respectively be equal to* A *and* B.

(1) If $\ell(A)+\ell(B) \le 2|\mathbb{K}|-5$, *then* $(d(A),d(B))=1$ *and*

$$|2\mathbb{K}| \ge (3|\mathbb{K}|-3)+h(A)+h(B) = (2|\mathbb{K}|-1)+\ell(A)+\ell(B).$$

(2) If $\ell(A)+\ell(B) \ge 2|\mathbb{K}|-4$ *and* $(d(A),d(B))=1$, *then* $|2\mathbb{K}| \ge 4|\mathbb{K}|-6$. □

It is not difficult to give examples to show that Theorems A(2), B(2) and Theorem S cannot be sharpened by reducing the quantity v or by increasing the upper bound for $|2\mathbb{K}|$. (see Examples B1 and B2 of Section 3, **[S2]**)

The present paper is devoted to the generalization of Theorem A(2) and S to the case of $s=3$ parallel lines. Instead of condition $|2\mathbb{K}|<3k-3$, of Theorem A and condition $|2\mathbb{K}|<\frac{10}{3}k-5$ of Theorem B, we study now a set $\mathbb{K}$ of integer points on a plane, with the following small doubling property

$$|2\mathbb{K}| < 3.5|\mathbb{K}|-7.$$

Take a lattice $\mathcal{L}$ generated by $\mathbb{K}$. We wish to obtain an estimate for the number of points of $\mathcal{L}$ that lie in $\mathrm{conv}(\mathbb{K})$; we are interested in an upper bound of $|\mathcal{L}\cap\mathrm{conv}(\mathbb{K})|$. Some estimate of this number was obtained in [S2, Theorem C]. In this paper we shall give the best possible estimate for $|\mathcal{L}\cap\mathrm{conv}(\mathbb{K})|$. The result implies an affirmative answer to a question of G.A. Freiman **[F3]** and generalizes previous results of **[F1]** and **[S2]**.

2. Main Result

An *arithmetic progression* in $\mathbb{Z}^2$ is a set of the form

$$P=P(a,\Delta)=\{a,a+\Delta,a+2\Delta,\dots,a+(p-1)\Delta\},$$

where $a,\ \Delta\in\mathbb{Z}^2$ and $p=|P|\ge 1$. The vector Δ is called the *common difference* of the progression and a is the *initial term* . We say that $P_i=P_i(a_i,\ \Delta_i)$, $i=1,2,3$ are *compatible* arithmetic progressions, if $\Delta_1=\Delta_2=\Delta_3=\Delta$ and $a_1+a_3\equiv 2a_2(\mathrm{mod}\,\Delta)$.

Now we are ready to formulate our main result.

Theorem 1. — *Let* $\mathbb{L}\subseteq\mathbb{Z}^2$ *be a finite set of lattice points with small doubling property:*

$$|\mathbb{L}+\mathbb{L}| < 3.5|\mathbb{L}|-7. \tag{2.1}$$

(1) If $|\mathbb{L}|\ge 1344$, *then the set* $\mathbb{L}$ *lies on no more than three parallel lines.*
(2) If $\mathbb{L}$ *is not contained in any two parallel lines, then* $\mathrm{conv}(\mathbb{L})\cap\mathbb{Z}^2$ *is included in three compatible arithmetic progressions having together no more than*

$$v=|\mathbb{L}|+\frac{3}{4}\Big(|\mathbb{L}+\mathbb{L}|-\frac{10}{3}|\mathbb{L}|+5\Big)=\frac{3}{4}\Big(|\mathbb{L}+\mathbb{L}|-2|\mathbb{L}|+5\Big) \tag{2.2}$$

terms.

Assertion (1) of Theorem 1 is a partial case of Theorem C, for $s=3$. We shall reformulate our main result and prove that the new formulation implies assertion (2)

of Theorem 1. We need some definitions. Let $\mathbb{K} \subseteq \mathbb{Z}^2$ be a finite set of lattice points that lies on three parallel lines:

$$\mathbb{K} = \mathbb{K}_1 \cup \mathbb{K}_2 \cup \mathbb{K}_3,$$
$$\mathbb{K}_1 \subseteq (x_2 = 0),\ \mathbb{K}_2 \subseteq (x_2 = 1),\ \mathbb{K}_3 \subseteq (x_2 = h),\ h \geq 2. \tag{2.3}$$

Let the set of abscissae of $\mathbb{K}_i$ be respectively equal to K_i and denote $d_i = d(K_i)$.

Put

$$\mathbb{K}^* = \mathrm{conv}(\mathbb{K}) \cap \mathbb{Z}^2,\ k = |K|,\ k^* = |\mathbb{K}^*| \tag{2.4}$$

and

$$d(\mathbb{K}) = \gcd(d_1, d_2, d_3). \tag{2.5}$$

Such a finite set of $\mathbb{Z}^2$ is called a *reduced set of lattice points,* if $h = 2$ and $d(\mathbb{K}) = 1$.

We would like to note at this point that this definition may be formulated in an obvious way, for sets that lie on $s \geq 2$ parallel lines. In this paper, however, a reduced set of lattice points will always be a set that lies on three parallel lines.

Theorem 2. — *Let $\mathbb{K} \subseteq \mathbb{Z}^2$ be a reduced set of lattice points. If $|2\mathbb{K}| < 3.5|\mathbb{K}| - 7$, then*

$$k^* := |\mathrm{conv}(\mathbb{K}) \cap \mathbb{Z}^2| \leq |\mathbb{K}| + \frac{3}{4}\Big(|2\mathbb{K}| - \frac{10}{3}|\mathbb{K}| + 5\Big) = \frac{3}{4}\Big(|2\mathbb{K}| - 2|\mathbb{K}| + 5\Big).$$

Proof of case (2) of Theorem 1, assuming Theorem 2. — Since $\mathbb{L}$ lies on three parallel lines, there is an affine isomorphism of the plane which maps $\mathbb{L}$ onto a set $\mathbb{K}$ such that

(i) $\mathbb{K}$ lies on $(x_2 = 0)$, $(x_2 = 1)$, $(x_3 = h)$, $h \geq 2$,

(ii) $m_1 = m_2 = 0$, where we put $m_i = \min(K_i)$, for $i = 1, 2, 3$.

Since the function $|2\mathbb{L}|$ is an affine invariant of the set $\mathbb{L}$, we see that

$$|2\mathbb{K}| = |2\mathbb{L}| < 3.5|\mathbb{L}| - 7 = 3.5|\mathbb{K}| - 7. \tag{2.6}$$

Denote $d = d(\mathbb{K})$. Remark that, thanks to the small doubling property (2.6) one has

$$h = 2 \text{ and } m_1 + m_3 \equiv 2m_2 (\mathrm{mod}\, d). \tag{2.7}$$

Indeed, if $h > 2$, then $(\mathbb{K}_1 + \mathbb{K}_3) \cap 2\mathbb{K}_2 = \varnothing$ and thus

$$\begin{aligned} |2\mathbb{K}| &\geq |2K_1| + |K_1 + K_2| + |K_1 + K_3| + |2K_2| + |K_2 + K_3| + |2K_3| \\ &\geq (2k_1 - 1) + (k_1 + k_2 - 1) + (k_1 + k_3 - 1) \\ &\quad + (2k_2 - 1) + (k_2 + k_3 - 1) + (2k_3 - 1) \\ &= 4k - 6 \geq 3.5k - 7. \end{aligned} \tag{2.8}$$

In the same way, if $m_1 + m_3 \not\equiv 2m_2 (\mathrm{mod}\, d)$, then for $x \in K_1; y', y'' \in K_2$, $z \in K_3$ we have $y' + y'' \equiv 2m_2 \not\equiv m_1 + m_3 = x + z (\mathrm{mod}\, d)$. Thus, $(\mathbb{K}_1 + \mathbb{K}_3) \cap 2\mathbb{K}_2 = \varnothing$ is valid and (2.8) follows again.

Consequently, $\mathbb{K}$ and $\mathbb{L}$ are contained each in three equidistant compatible arithmetic progressions.

Equation (2.7) and (ii) ensure that $m_3 \equiv 2m_2 - m_1 = 0 (\mathrm{mod}\, d)$. This yields $w \equiv 0 (\mathrm{mod}\, d)$ for every $w \in K_1 \cup K_2 \cup K_3$. We can now easily check that the