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NON-SOLVABLE GROUPS WITH A LARGE FRACTION OF INVOLUTIONS

by

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Abstract. — In this note we classify the non-solvable finite groups G such that the class number of G is at least $|G|/16$. Some consequences are derived as well.

C.T.C. Wall classified all finite groups in which the fraction of involutions exceeds 1/2 (see **[1]**, Theorem 11.24). In this paper we classify all non-solvable finite groups in which the fraction of involutions is not less than 1/4.

We recall some notation.

Let $k(G)$ be the class number of G. Let $i(G)$ denote the number of all involutions of G, $T(G) = \sum \chi(1)$ where χ runs over the set $\mathrm{Irr}(G)$. Now

$$\mathrm{mc}(G) = k(G)/|G|, \ f(G) = T(G)/|G|, \ i_o(G) = i(G)/|G|.$$

It is well-known (see **[1]**, chapter 11) that

$$i(G) < T(G), \ i_o(G) < f(G), \ f(G)^2 \leq \mathrm{mc}(G)$$

(with equality if and only if G is abelian).

In this note we prove the following three theorems.

Theorem 1. — *Let G be a non-solvable group.*

If $mc(G) \geq 1/16$ then $G = G'Z(G)$, where G' is the commutator subgroup of G, $Z(G)$ is the centre of G, $G' \in \{PSL(2,5), SL(2,5)\}$.

Theorem 2. — *Let G be a non-solvable group.*

If $f(G) \geq 1/4$ then $G = G'Z(G)$ and $G' \in \{PSL(2,5), SL(2,5)\}$.

Theorem 3. — *Let G be a non-solvable group.*

Then $i_o(G) \geq 1/4$ if and only if $G = PSL(2,5) \times E$ with $\exp E \leq 2$.

Lemma 1 contains some well-known results.

Lemma 1

(a) If G is simple and a non-linear $\chi \in Irr(G)$ is such that $\chi(1) < 4$, then $\chi(1) = 3$ and $G \in \{PSL(2,5), PSL(2,7)\}$; see [2].

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(b) *(Isaacs; see* **[1]***, Theorem 14.19). If G is non-solvable, then $|cdG| \geq 4$; here $cdG = \{\chi(1) | \chi \in Irr(G)\}$.*

(c) *(see, for example,* **[1]***, Chapter 11). If G is non-abelian then*

$$mc(G) \leq 5/8, \ f(G) \leq 3/4.$$

Lemma 2. — *Let $G = G' > 1$, $d \in \{4,5,6\}$. If $mc(G) \geq (1/d)^2$ then there exists a non-linear $\chi \in Irr(G)$ such that $\chi(1) < d$.*

Proof. — Suppose that G is a counterexample. Then by virtue of Lemma 1(b) one has

$$\begin{aligned} |G| &= \textstyle\sum_\chi \chi(1)^2 \geq 1 + d^2(k(G) - 3) + (d+1)^2 + (d+2)^2 \\ &\geq 1 + d^2(\tfrac{|G|}{d^2} - 3) + 2d^2 + 6d + 5 = |G| - d^2 + 6d + 6 > |G| \end{aligned}$$

since $d \in \{4,5,6\}$, — a contradiction (here χ runs over the set $\mathrm{Irr}(G)$).

Lemma 3 contains the complete classification of all groups G satisfying $i_o(G) = 1/4$.

Lemma 3. — *If $i_o(G) = 1/4$ then one and only one of the following assertions holds:*

(a) *$G \cong A_4$, the alternating group of degree 4.*

(b) *$G \cong PSL(2,5)$.*

(c) *G is a Frobenius group with kernel of index 4.*

(d) *G is a non-cyclic abelian group of order 12.*

(e) *G contains a normal subgroup R of order 3 such that $G/R \cong S_3 \times S_3$; if x is an involution in G then $|C_G(x)| = 12$ (here S_3 is the symmetric group of degree 3).*

Proof. — By the assumption $|G|$ is even. $i(G)$ is therefore odd by the Sylow Theorem and $|G| = 4i(G)$, $P \in \mathrm{Syl}_2(G)$ has order 4.

(i) Suppose that G has no a normal 2-complement. Then P is abelian of type $(2,2)$ and by the Frobenius normal p-complement Theorem G contains a minimal non-nilpotent subgroup $F = C(3^a) \cdot P$ (here $C(m)$ is a cyclic group of order m and $A \cdot B$ is a semi-direct product of A and B with kernel B). Since all involutions are conjugate in F, all involutions are conjugate in G. Hence $C_G(x) = P$ for $x \in P^\# = P - \{1\}$, $a = 1$. If G is simple then by the Brauer-Suzuki-Wall Theorem (see **[1]**, Theorem 5.20) one has

$$|G| = (2^2 - 1)2^2(2^2 + 1) = 60.$$

Now we assume that G is not simple. Take H, a non-trivial normal subgroup of G. If $|G : H|$ is odd, then

$$\begin{aligned} i(G) = i(H),\ i_o(H) = i(H)/|H| = i(G)/|H| = \\ |G| i_o(G)/|H| = |G : H| i_o(G) = |G : H|/4. \end{aligned}$$

Therefore $|G : H| = 3$ and $i_o(H) = 3/4$. Now $f(H) > i_o(H)$, hence H is abelian (Lemma 1(c)) and $f(H) = 1$. It is easy to see that H is an elementary abelian 2-group, $H = P$. Now $|P| = 4$ implies $|G| = 12$, $F = G \cong A_4$.

Now suppose that H has even index. Since G is not 2-nilpotent (= has no a normal 2-complement) then $|H|$ is odd. In view of $|C_G(x)| = 4$ for $x \in P^\#$ one obtains that PH is a Frobenius group with kernel H, P is cyclic — a contradiction.

(ii) G has a normal 2-complement K.

First assume that P is cyclic. Then all involutions are conjugate in G, and for the involution $x \in P$ one has $C_G(x) = P$. Then G is a Frobenius group with kernel K of index 4.

Assume that $P = \langle\alpha\rangle \times \langle\beta\rangle$ is not cyclic. We have $P = \{1, \alpha, \beta, \alpha\beta\}$, and all elements from $P^{\#}$ are not pairwise conjugate in G. Thus

$$|G : C_G(\alpha)| + |G : C_G(\beta)| + |G : C_G(\alpha\beta)| = i(G) = |G : P|.$$

Note that $C_G(\alpha) = P \cdot C_K(\alpha)$, and similarly for β and $\alpha\beta$. Therefore

$$|C_K(\alpha)|^{-1} + |C_K(\beta)|^{-1} + |C_K(\alpha\beta)|^{-1} = 1. \tag{1}$$

Since $|K| > 1$ is odd then (1) implies

$$|C_K(\alpha)| = |C_K(\beta)| = |C_K(\alpha\beta)| = 3. \tag{2}$$

By the Brauer Formula (see **[1]**, Theorem 15.47) one has

$$|K||C_K(P)|^2 = |C_K(\alpha)||C_K(\beta)||C_K(\alpha\beta)| = 3^3. \tag{3}$$

If $C_K(P) > 1$ then (3) implies $|K| = 3$ and $G = P \times K$ is an abelian non-cyclic group of order 12.

Assume $C_K(P) = 1$. Then $|K| = 3^3$. Now (2) implies that K is not cyclic. By analogy, (2) implies that $\exp K = 3$. From $\exp P = 2$ follows that G is supersolvable. Therefore R, a minimal normal subgroup of G, has order 3. Applying the Brauer Formula to G/R, one obtains $G/R \cong S_3 \times S_3$, and we obtain group (e).

Proof of Theorem 1. — Denote by $S = S(G)$ the maximal normal solvable subgroup of G.

(i) If G is non-abelian simple then $G \cong \mathrm{PSL}(2,5)$.

Proof. — Take $d = 4$ in Lemma 2. Then there exists $\chi \in \mathrm{Irr}(G)$ with $\chi(1) = 3$. Now Lemma 1(a) implies $G \in \{\mathrm{PSL}(2,5), \mathrm{PSL}(2,7)\}$. Since

$$\mathrm{mc}(\mathrm{PSL}(2,7)) = 1/28 < 1/16$$

then $G \cong \mathrm{PSL}(2,5)$ (note that $\mathrm{mc}(\mathrm{PSL}(2,5)) = 1/12$).

(ii) If G is semi-simple then $G \cong \mathrm{PSL}(2,5)$.

Proof. — Take in G a minimal normal subgroup D. Then $D = D_1 \times \cdots \times D_s$ where the D_i's are isomorphic non-abelian simple groups. Since (see **[1]**, Chapter 11) $\mathrm{mc}(D_1) \geq \mathrm{mc}(G) \geq 1/16$, $D \cong \mathrm{PSL}(2,5)$ by (i) and so $\mathrm{mc}(D_1) = 1/12$. Now

$$\mathrm{mc}(D) = \mathrm{mc}(D_1)^s = (1/12)^s \geq 1/16$$

implies that $s = 1$. Therefore $D \cong \mathrm{PSL}(2,5)$. Since $G/C_G(D)$ is isomorphic to a subgroup of $\mathrm{Aut}D \cong S_5$, $\mathrm{mc}(S_5) = 7/120 < 1/16$, then $G/C_G(D) \cong \mathrm{PSL}(2,5)$. Because $D \cap C_G(D) = 1$, $G = D \times C_G(D)$. Now

$$1/16 \leq \mathrm{mc}(G) = \mathrm{mc}(C_G(D))\mathrm{mc}(D) = (1/12)\mathrm{mc}(C_G(D))$$

implies that $\mathrm{mc}(C_G(D)) \geq 3/4 > 5/8$, $C_G(D)$ is abelian (Lemma 1(c)), $C_G(D) = 1$ (since G is semi-simple), and $G \cong \mathrm{PSL}(2,5)$.

(iii) $G/S \cong \mathrm{PSL}(2,5)$.

This follows from $\mathrm{mc}(G/S) \geq \mathrm{mc}(G)$ (P.Gallagher; see **[1]**, Theorem 7.46) and (ii).

(iv) If $G = G'$ then $G \in \mathrm{PSL}(2,5), \mathrm{SL}(2,5)\}$.

Proof. — By virtue of (iii) we may assume that $S > 1$.

Suppose that (iv) is true for all proper epimorphic images of G. Take in S a minimal normal subgroup R of G, and put $|R| = p^n$. Then by the Gallagher Theorem and induction one has $G/R \in \{PSL(2,5), \mathrm{SL}(2,5)\}$.

(1iv) $G/R \cong \mathrm{PSL}(2,5)$, i.e. $R = S$.

If $Z(G) > 1$ then $R = Z(G)$ is isomorphic to a subgroup of the Schur multiplier of G/R so $|R| = 2$ and $G \cong \mathrm{SL}(2,5)$ (Schur). In the sequel we suppose that $Z(G) = 1$.

Then $C_G(R) = R$, so $n > 1$. If $x \in R^\#$ then $|G : C_G(x)| \geq 5$, since index of any proper subgroup of $\mathrm{PSL}(2,5)$ is at least 5. Let $k_G(M)$ denote the number of conjugacy classes of G ($= G$-classes), containing elements from M. Then

$$k_G(R) \leq 1 + |R^\#|/5 = (p^n + 4)/5.$$

If $x \in G - R$ then $Z(G) = 1$, and the structure of G/R imply $|G : C_G(x)| \geq 12p$ (indeed, x does not centralize R and $|G/R : C_{G/R}(xR)| \geq 12$). Hence

$$k_G(G - R) = k(G) - k_G(R) = |G|\mathrm{mc}(G) - k_G(R) \geq \\ 60p^n/16 - (p^n + 4)/5 = (71p^n - 16)/20.$$

Now

$$|G - R| = 59p^n \geq 12pk_G(G - R) \geq 12p(71p^n - 16)/20, \tag{1}$$

$$5 \times 59p^{n-1} = 295p^{n-1} \geq 213p^n - 48 \geq 426p^{n-1} - 48 \Rightarrow 131p^{n-1} \leq 48, \tag{2}$$

a contradiction.

(2iv) $G/R \cong \mathrm{SL}(2,5)$.

Proof. — Suppose that $R_1 \neq R$ is a minimal normal subgroup of G. Then (by induction)

$$RR_1 = R \times R_1 = S,\ |R_1| = 2,\ G/R_1 \cong \mathrm{SL}(2,5)$$

and $G' < G$, since the multiplier of $\mathrm{SL}(2,5)$ is trivial, a contradiction. Therefore R is a unique minimal normal subgroup of G. Similarly, one obtains $Z(G) = 1$.

Let $p > 2$. Then $C_G(R) = R$. In this case $Z(S) < R$, so $Z(S) = 1$ and S is a Frobenius group with kernel R of index 2. As in (1iv) one has

$$k_G(S) = k_G(S - R) + k_G(R) \leq 1 + (p^n + 4)/5 = (p^n + 9)/5.$$

If $x \in G - S$ then $|G : C_G(x)| \geq 12p$ and

$$\begin{aligned} k_G(G - S) = k(G) - k_G(S) = |G|\mathrm{mc}(G) - k_G(S) \geq \\ 120p^n/16 - (p^n + 9)/5 = (73p^n - 18)/10, \\ |G - S| = 118p^n \geq 12pk_G(G - S) \geq 6p(73p^n - 18)/5, \\ 295p^{n-1} \geq 219p^n - 54 \geq 657p^{n-1} - 54, \\ 54 \geq 362p^{n-1}, \end{aligned}$$