Astérisque

MARK CHAIMOVICH

New structural approach to integer programming : a survey

Astérisque, tome 258 (1999), p. 341-362
<http://www.numdam.org/item?id=AST_1999__258__341_0>

Astérisque **258**, 1999, p. 341–362

NEW STRUCTURAL APPROACH TO INTEGER PROGRAMMING: A SURVEY

by

Mark Chaimovich

Abstract. — The survey discusses a new approach to Integer Programming which is based on the structural characterization of problems using methods of additive number theory. This structural characterization allows one to design algorithms which are applicable in a narrower, yet still wide, domain of problems, and substantially improve the time boundary of existing algorithms. The new algorithms are polynomial for the class of problems in which they are applicable, and even linear ($O(m)$) for a wide class of the Subset-Sum and Value-Independent Knapsack problems. Previously known polynomial time algorithms for the same classes of problems are at least two orders of magnitude slower.

1. Introduction

This survey considers a recently developed approach to Integer Programming (IP) which is based on the application of analytical methods of Additive Number Theory. Elaborated by G. Freiman in the early 1980's, this new approach was developed by N. Alon, P. Buzytsky, M. Chaimovich, P. Erdős, G. Freiman, Z. Galil, E. Lipkin and O. Margalit (in alphabetical order).

In general, the number of Integer Programming models is vast and they have numerous applications; only a few of them – Subset-Sum (one and multi-dimensional), Value-Independent Knapsack and k-Partition problems – were investigated using the new structural approach. Theorems from analytical number theory allow one to characterize the structure of the domain of solutions for a wide class of problems and to design efficient algorithms for these problems. These new algorithms substantially improve the time boundary of existing algorithms. They are polynomial for the class of problems in which they are applicable, and even linear ($O(m)$) for certain classes of the Subset-Sum and Value-Independent Knapsack problems. That is at least two orders of magnitude faster than previously known polynomial time algorithms for the

1991 ***Mathematics Subject Classification.*** — Primary: 90C02 Alternate: 05A17, 11B25, 68Q25.
Key words and phrases. — Analytical Number Theory, Integer Programming, Subset Sum Problem.

same classes of problems. This fact allows one to solve problems with a much larger number of variables.

This article is organized into several parts. In section 2 the general idea for development of an analytical approach to Integer Programming is considered. Sections 3 and 4 deal with the Subset-Sum Problem (SSP). The first of them provides a detailed, structural analysis of the problem including an example of the analytical theorem while the second describes algorithms for solving SSP based on this structural analysis. Proofs of the validity of the algorithms are not provided in this survey, however, they may be found in the references. Section 5 describes the application of the structural approach to multi-dimensional Subset-Sum, Value-Independent Knapsack and k-Partition problems. (Only the main theorems and outlines of the algorithms are presented.) In the conclusion possible directions for future research are discussed.

2. General idea of the application of the structural approach to IP

In this section the main idea of the structural approach is described. We begin with a simple example that illustrates the approach. Further, the concept of density is discussed, this explains how the structural characterization of the problem may be obtained. We conclude the section with a short history of the research in the field of structural characterization.

2.1. A simple illustration of the structural approach. — In order to understand a structural approach to IP, consider the problem of feasibility of a single boolean equation. Given an integer m, an integral vector $(a_1, a_2, \dots, a_m)$ and an integer N, does equation

$$a_1x_1 + a_2x_2 + \cdots + a_mx_m = N \tag{1}$$

have any solutions for $x_i \in \{0,1\}$ for all i? To illustrate the approach, we use the following concrete equation

$$7x_1 + 8x_2 + 14x_3 + 15x_4 + 22x_5 + 28x_6 + 56x_7 = 75, \tag{2}$$

i.e. $m = 7$, $(a_1, \dots, a_7) = (7, 8, 14, 15, 22, 28, 56)$ and $N = 75$.

Dynamic programming approach

Denoting $S_0 = \{0\}$ and $S_k = \{b | b = \sum_{j=1}^{k} a_jx_j, x_j \in \{0,1\}\}$ for $1 \le k \le 7$, we have $S_k = S_{k-1} + \{0, a_k\} = \{b | b \in S_{k-1} \text{ or } b - a_k \in S_{k-1}\}$. Thus, having S_7 – the set of all possible values of the linear form in the left-hand side of (2),– it remains only to check if $N = 75 \in S_7$. In fact,

$$\begin{aligned} S_1 &= \{0, 7\}, \\ S_2 &= \{0, 7, 8, 15\}, \\ S_3 &= \{0, 7, 8, 14, 15, 21, 22, 29\}, \\ &\dots \end{aligned}$$

and so on. Finally,

$$\begin{aligned} S_7 &= \{0,7,8,14,15,21,22,23,28,29,30,35,36,37,42,43,44,45,49,50, \\ &\quad 51,52,56,57,58,59,63,64,65,66,70,71,72,73,77,78,79,80,\dots\}, \end{aligned}$$

i.e., $75 \notin S_7$ and equation (2) does not have a solution.

Structural approach

We characterize the structure of S_7 without explicitly enumerating it. Observe, that some of the coefficients of the equation are divisible by 7: $a_1 \equiv a_3 \equiv a_6 \equiv a_7 \equiv 0 (\mathrm{mod}\, 7)$. Then, for $b \in S_7$ we have $b \equiv 8x_2 + 15x_4 + 22x_5 \equiv x_2 + x_4 + x_5 (\mathrm{mod}\, 7)$, i.e.,

$$b \equiv 0, 1, 2, 3 (\mathrm{mod}\, 7). \tag{3}$$

However, $75 \equiv 5(\mathrm{mod}\, 7)$, so, the equation does not have a solution.

Condition (3) determines a necessary condition for solvability equation (2). In order to obtain a sufficient condition let us analyze the same equation with another right-hand side:

$$7x_1 + 8x_2 + 14x_3 + 15x_4 + 22x_5 + 28x_6 + 56x_7 = 79.$$

Clearly, $79 \equiv 2(\mathrm{mod}\, 7)$, so it can belong to S_7 according to (3). To confirm that it really belongs to S_7, consider a linear form

$$L = 7x_1 + 14x_3 + 28x_6 + 56x_7 = 7(x_1 + 2x_3 + 4x_6 + 8x_7).$$

The linear form $L' = x_1 + 2x_3 + 4x_6 + 8x_7$ can take all values from 0 to 15, thus, the linear form L can, correspondingly, take values of the form $7t$, where $0 \le t \le 15$. When we combine these values with the other coefficients $(8, 15, 22)$, we have

$$\begin{aligned} S_7 = \{b \mid\ & b \equiv 0(\mathrm{mod}\, 7), 0 \le b \le 7 \cdot 15, \ \textit{or} \\ & b \equiv 1(\mathrm{mod}\, 7), 8 \le b \le 22 + 7 \cdot 15, \ \textit{or} \\ & b \equiv 2(\mathrm{mod}\, 7), 23 \le b \le 37 + 7 \cdot 15, \ \textit{or} \\ & b \equiv 3(\mathrm{mod}\, 7), 45 \le b \le 45 + 7 \cdot 15\}. \end{aligned} \tag{4}$$

Here 8, 23, 45 are the smallest numbers with residues 1, 2, 3 modulo 7 that can be represented by the linear form in the left-hand side of the equation. Since $79 \equiv 2(\mathrm{mod}\, 7)$ and $79 = 23 + 7 \cdot 8$, the answer is that the equation has at least one solution.

Observe that the above consideration determines the structure of the set of possible values of a linear form on the left-hand side of an equation as a collection of arithmetic progressions with a common difference. This fact allows one to solve the problem immediately for each right-hand side. One can suppose that this example was especially selected to illustrate the approach and that would be true. However the situation obtained can be generalized: for a wide class of problems we can always determine the structure.

To obtain a general structural characterization of the IP problem (in the same way that (4) was obtained for a concrete equation), a specific analytical theorem must be proven. Of course, certain conditions have to be imposed on the coefficients in order to obtain such a characterization. These conditions follow directly from the analytical theorem. Once we have the conditions, it is possible to go to the next step – to design algorithms to verify these conditions and to obtain the structure.

Indeed, the structure obtained and the conditions of its existence provide an understanding of why some problems are easy and others are very hard for various enumerative algorithms. To confirm this statement consider the following problem which was investigated by R. Jeroslow (1974) **[19]**: maximize x_1 satisfying $2x_1 + 2x_2 + \cdots + 2x_n = n$ where n is odd. Although this problem is by nature trivial, it requires almost complete enumeration using different enumerative techniques. (Branch and Bound, for example, is one of them.) The secret is the fact that the constraint has no solutions, however, we must verify all possibilities to confirm this fact. The structural approach allows one to obtain an answer for this problem in no time.

2.2. Concept of density and its use in structural characterization. — In order to apply analytical methods to solve an IP problem, it is necessary for the problem to have a *high density.* To explain the notion of "density" and its importance in the application of the analytical approach to IP, let us consider again the feasibility of equation (1).

Let $\ell = \max_{1\leq i\leq m} a_i$. The linear function on the left in (1) has a domain of size 2^m and a range of size $m\ell$. Since the domain size represents the overall number of "solutions" for all possible values of the right-hand side, the ratio $\frac{2^m}{m\ell}$ represents the average number of "solutions" for a value from the range. We say that this ratio characterizes the density of the problem. The density of other IP problems can be defined similarly.

In the case of equation (1), the density condition means that $\ell = o(\frac{2^m}{m})$ or $\frac{2^m}{m\ell} \to \infty$. Currently, algorithms are still not capable of handling this density. The only situation that has been investigated is $\ell = O(\frac{m^2}{\log m})$. The conjecture of G. Freiman is that the new approach can be refined to handle the case $\ell = O(m^c)$ for any positive constant c.

To highlight basic features of the approach, we present some non-strict considerations resulting from probability theory. In view of

$$\int_0^1 e^{2\pi i\alpha b} d\alpha = \begin{cases} 0 & \text{for } b \in \mathbb{Z}, b \neq 0, \\ 1 & \text{for } b = 0, \end{cases}$$

it is easy to verify that the number of solutions of (1) can be expressed by the integral

$$J(N) = \int_0^1 \prod_{j=1}^{m}(1 + e^{2\pi i\alpha a_j})e^{-2\pi i\alpha N} d\alpha = 2^m \int_0^1 \prod_{j=1}^{m}(\tfrac{1}{2} + \tfrac{1}{2}e^{2\pi i\alpha a_j})e^{-2\pi i\alpha N} d\alpha. \tag{5}$$

One may look at $\frac{1}{2} + \frac{1}{2}e^{2\pi i\alpha a_j}$ as the characteristic function of a random variable ξ_j taking values 0 and a_j with probabilities equal to $\frac{1}{2}$. Then the value of integral (5) is equal to the probability $P(\zeta = N)$, where $\zeta = \xi_1 + \cdots + \xi_m$ is a random variable with mathematical expectation $M = \frac{1}{2}\sum_{j=1}^{m} a_j$ and dispersion $\sigma^2 = \frac{1}{4}\sum_{j=1}^{m} a_j^2$. Assuming that the local limit theorem can be applied, the variable ζ has asymptotically normal