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SETS OF INTEGERS WITH LARGE TRIGONOMETRIC SUMS

by

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Abstract. — We investigate the problem of optimizing, for a fixed integer k and real u and on all sets $K = \{a_1 < a_2 < \cdots < a_k\} \subset \mathbb{Z}$, the measure of the set of $\alpha \in [0,1]$ where the absolute value of the trigonometric sum $S_K(\alpha) = \sum_{j=1}^{k} e^{2\pi i\alpha a_j}$ is greater than $k - u$. When u is sufficiently small with respect to k we are able to construct a set K_{ex} which is very close to optimal. This set is a union of a finite number of arithmetic progressions. We are able to show that any more optimal set, if one exists, has a similar structure to that of K_{ex}. We also get tight upper and lower bounds on the maximal measure.

1. Introduction

Let k be a positive integer and $u < k$ a positive real. For a set

$$K = \{a_1 < a_2 < \cdots < a_k\}, \quad a_j \in \mathbb{Z}, \quad 1 \le j \le k,$$

let

$$S_K(\alpha) = \sum_{j=1}^{k} e^{2\pi i\alpha a_j}, \quad s_K(\alpha) = |S_K(\alpha)|,$$

$$E_{K,u} = \{\alpha \in [0,1) : \quad s_K(\alpha) \ge k - u\}$$

and

$$\mu_K(u) = \mu(E_{K,u}),$$

where μ is the Lebesgue measure on $[0,1]$ normalized so $\mu([0,1]) = 1$.

This work deals with the following problem, first raised at the talk of Freiman and Yudin at the Number Theory Conference (Vladimir, 1968):

Problem 1. — *Find the set K which maximizes $\mu_K(u)$ and find the maximal value.*

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We denote by $\mu_{\max}(k,u)$ the supremum of $\mu_K(u)$ on all sets K of size k.

The first results on this problem were obtained by Freiman and Yudin:

***Theorem 1* (Freiman, [2, page 144]).** — *For $u = 1$, $a_1 = 0$ and $a_k < 0.05k^{3/2}$, the maximal measure is*

$$\mu_{\max}(k,u) = \frac{2\sqrt{6}}{\pi}k^{-3/2} + O(k^{-2})$$

and it is attained by K if and only if K is an arithmetic progression.

***Theorem 2* (Yudin, [4]).** — *For $u = o(k)$*

$$\mu_{\max}(k,u) = \frac{2\sqrt{6}}{\pi}\frac{1}{k}\left(\frac{u}{k}\right)^{1/2}(1+o(1))$$

as $k \longrightarrow \infty$.

In [1] Freiman treated the problem assuming the ratio u/k is small enough. He sketched an approach for attacking the problem and conjectured it would prove that the best set is an arithmetic progression.

The purpose of this work is to carry out and extend Freiman's approach (and also that of [4]). It will turn out that once u is sufficiently large it is no longer true that an arithmetic progression attains $\mu_{\max}(k,u)$. We are unable to find a set which does. Nevertheless, we do describe to some extent the "structure" of the maximal set. To make precise what this means, we will introduce and use the following terminology

Definition 1.1. — Let k and u be as above.

1. For any $\psi \in [0,1]$ we let $\mathbb{K}_\psi$ be the collection of all sets $K \subset \mathbb{Z}$ of size k that satisfy $\mu_K(u) \geq \psi$.
2. A collection $\mathbb{K}$ of sets is said to be "good for ψ", or to be a G_ψ collection, if it satisfies the following two properties:
   (a) We have $\mathbb{K} \subset \mathbb{K}_\psi$,
   (b) For any set $K \subset \mathbb{Z}$ of size k there exists a set $K' \in \mathbb{K}$ such that $\mu_K(u) \leq \mu_{K'}(u)$.

Our main results are of three types. We are able to describe the "structure" of sets in $\mathbb{K}_\psi$ for a ψ which is very close to maximal. In addition we construct a certain sub-collection of this which has property G_ψ. The subclass we describe is not a singleton but it does have a rather simple structure: it is essentially the union of arithmetic progressions, and we have a fairly accurate information about the location and length of all these sequences. Lastly, we get a good bound on $\mu_{\max}(k,u)$.

The type of results we get is dictated by our method of proof, which could be describe as an iteration of four steps.

1. We first guess a set K expected to have a large $\mu_K(u)$. We will take $\psi = \mu_K(u)$.
2. We get information about sets K_1 that have an even higher $\mu_{K_1}(u)$. Typically this information consists of knowledge that most elements are contained inside arithmetic progressions of relatively short length. This is what we mean by describing the "structure" of $\mathbb{K}_\psi$.

3. Given a set $K_1 \in \mathbb{K}_\psi$, we give a procedure for obtaining out of K_1 a set K_2 such that $\mu_{K_2}(u) \geq \mu_{K_1}(u)$. The procedure usually involves compressing the elements contained in the short progressions described above to form short progressions, possibly with a single gap. Sets obtained in this way will form a subclass $\mathbb{K}$ that has property G_ψ by construction.
4. We use the knowledge of $\mathbb{K}_\psi$ to get an improved bound on $\mu_{\max}(k,u)$.

Our results apply under the assumptions $k/30000 \geq u > 1$ and $k \geq$ Const, with Const an unspecified constant. We note that this second assumption is only forced on us because we are using lemma 5.2 which is ineffective. If an effective bound is supplied for that lemma, it will be very easy to deduce an effective lower bound on k as well.

We state a simplified version of the results here under the additional assumption $u > k^{2/3}$. With these assumptions, it follows (proposition 3.1) that for an arithmetic progression K of difference 1 and length k there exists some $\beta_{k,u}$(definition 3.2) such that

$$E_{K,u} = [-\beta_{k,u}, \beta_{k,u}] \pmod 1.$$

We will see in proposition 3.4 that

$$\beta_{k,u} \approx \frac{\sqrt{6}}{\pi}\frac{1}{k}\left(\frac{u}{k}\right)^{1/2}.$$

We describe a certain basic set K_{ex} (a more precise description will be given in construction 6.1). Set $m_0 = k - 5u/12$ and $\beta = \beta_{m_0,u}$. To first order, $\beta_{m_0,u} \approx \beta_{k,u}(1 + 5u/8k)$. The set K_{ex} is the union of an arithmetic progression of length m_0, symmetric around 0, and for any non zero integer n an arithmetic progression of length

$$\frac{1}{2}m_n = \frac{u}{(\pi n)^2}\left(1 - \frac{(-1)^n}{2}\right),$$

centered around $\frac{n}{\beta}$. All the arithmetic progressions here have difference 1. The structure of K_{ex} and the particular values of the m_n are chosen in such a way that the contributions of the shorter progressions to $S_{K_{ex}}(\alpha)$ exactly compensates for the decline of the contribution of the large progression when $|\alpha| > \beta/2$. We show in proposition 6.12 that $\mu_{K_{ex}}(u) \approx 2\beta$. The results are now as follows.

1. $$\mu_{\max}(k,u) = 2\beta_{m_0,u}\left(1 + O\left(\left(\frac{u}{k}\right)^2\right)\right).$$

2. A set $K \in \mathbb{K}_{\mu_{K_{ex}}(u)}$ has the following structure (similar to that of K_{ex}).
   (a) All but
   $$\frac{5}{12}u + O\left(\frac{u^2}{k}\right)$$
   elements of K are contained in a short arithmetic progression of length $1/4\beta$. To state the other results we will assume that this progression is

symmetric around 0 and its difference is 1. The general case is essentially the same by translation and dilation which change nothing.

(b) Most other elements are contained in a union of short arithmetic progressions with centres near $\frac{n}{\beta}$ and $-\frac{n}{\beta}$ for $n \in \mathbb{N}$. each such short progression has length $2k$ at the most. The number of elements contained in progressions near $\pm\frac{n}{\beta}$ is

$$m_n + O\left(\frac{u^2}{k}\right).$$

(c) The number of elements not contained in any of the progressions above is $O\left((u/k)^{1/2}u\right)$.

3. The following subclass of $\mathbb{K}_{\mu_{K_{ex}}(u)}$ is of type $G_{\mu_{K_{ex}}(u)}$. It consists of the sets where all the elements *contained* in the progression described in (a) above in fact *form* an arithmetic progression except that one gap may persist.

All O terms can and will be made explicit although no claim for best bounds is made.

Here is a brief summary of the contents of this paper. In section 3 we estimate $\mu(E_{K,u})$ in the case where K is an arithmetic progression, and we prove the lower bound:

$$\mu(E_{K,u}) \leq \frac{2\sqrt{6}}{\pi}\frac{1}{k}\left(\frac{u}{k}\right)^{1/2}$$

for such a progression. In section 4 we prove the upper bound

$$\mu(E_{K,u}) \leq \frac{d}{k}\left(\frac{u}{k}\right)^{1/2}$$

holds for an explicitly given $d \approx 4$ and all sets K under some mild restrictions on k and u. In section 5 we consider a set

$$K \in \mathbb{K}_{\psi}, \quad \psi = \frac{2\sqrt{6}}{\pi}\frac{1}{k}\left(\frac{u}{k}\right)^{1/2}.$$

We show that this implies that $E_{K,u}$ is contained in a union of small intervals and that K has most of its elements contained in an arithmetic progression of short length. We then perform the first of our "compression arguments" mentioned above and construct a G_{ψ} subclass consisting of the sets where these elements form an arithmetic progression with at most one gap. The construction of K_{ex} is described in section 6. In sections 7 to 9 we describe the structure of $\mathbb{K}_{\mu_{K_{ex}}(u)}$ and also describe a $G_{\mu_{K_{ex}}(u)}$ subclass.

Some of the results of this paper appeared in [**1**]. We follow [**1**] very closely in sections 3 to 5. We note that the argument in [**1**, p. 368] may be completed to give the result that, the part of K not in an arithmetic progression is bounded in size by cu with $c \longrightarrow 1/(2-4/\pi)$ as $u/k \longrightarrow 0$. This result is improved here to $c \approx 5/12$.

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