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# ASYMPTOTIC MEASURES FOR HYPERBOLIC PIECEWISE SMOOTH MAPPINGS OF A RECTANGLE

*by*

Michael Jakobson & Sheldon Newhouse

*To Adrien Douady on the occasion of his sixtieth birthday*

***Abstract.*** — We prove the existence of Sinai-Ruelle-Bowen measures for a class of $C^2$ self-mappings of a rectangle with unbounded derivatives. The results can be regarded as a generalization of a well-known one dimensional Folklore Theorem on the existence of absolutely continuous invariant measures. In an earlier paper [8] analogous results were stated and the proofs were sketched for the case of invertible systems. Here we give complete proofs in the more general case of noninvertible systems, and, in particular, develop the theory of stable and unstable manifolds for maps with unbounded derivatives.

## 1. Folklore Theorem and SRB Measures

A well-known Folklore Theorem in one-dimensional dynamics can be formulated as follows.

***Folklore Theorem***. — *Let $I = [0,1]$ be the unit interval, and suppose $\{I_1, I_2, \dots\}$ is a countable collection of disjoint open subintervals of $I$ such that $\bigcup_i I_i$ has the full Lebesgue measure in $I$. Suppose there are constants $K_0 > 1$ and $K_1 > 0$ and mappings $f_i : I_i \to I$ satisfying the following conditions.*

(1) *$f_i$ extends to a $C^2$ diffeomorphism from* Closure$(I_i)$ *onto* $[0,1]$, *and*

$$\inf_{z \in I_i} |Df_i(z)| > K_0 \quad \textit{for all } i.$$

(2) $\sup_{z \in I_i} \dfrac{|D^2 f_i(z)|}{|Df_i(z)|} |I_i| < K_1$ *for all $i$.*

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*where $|I_i|$ denotes the length of $I_i$. Then, the mapping $F(z)$ defined by $F(z) = f_i(z)$ for $z \in I_i$, has a unique invariant ergodic probability measure $\mu$ equivalent to Lebesgue measure on $I$.*

For the proof of the Folklore theorem and the ergodic properties of $\mu$ see for example **[2]** and **[14]**.

In an earlier paper **[8]** we presented an analog of this theorem for piecewise $C^2$ diffeomorphisms with unbounded derivatives with proof sketched. We now wish to give a more general version of the results in **[8]**. We refer the reader to that paper for relevant remarks and references.

Let $\widetilde{Q}$ be a Borel subset of the unit square $Q$ in the plane $\mathbf{R}^2$ with positive Lebesgue measure, and let $F : \widetilde{Q} \to \widetilde{Q}$ be a Borel measurable map. An $F$-invariant Borel probablility measure $\mu$ on $Q$ is called a *Sinai – Ruelle – Bowen* measure (or SRB-measure) for $F$ if $\mu$ is ergodic and there is a set $A \subset \widetilde{Q}$ of positive Lebesgue measure such that for $x \in A$ and any continuous real-valued function $\phi : Q \to \mathbf{R}$, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \phi(F^k x) = \int \phi d\mu. \tag{1}$$

The set of all points $x$ for which (1) holds is called the *basin* of $\mu$.

Note that if $\mu$ is an SRB measure, and $m_1$ is the normalized Lebesgue measure on its basin, then the bounded convergence theorem gives the weak convergence of the averages $\frac{1}{n}\sum_{k=0}^{n-1} F_\star^k m_1$ of the iterates of $m_1$ to $\mu$. Hence, SRB measures occur as limiting mass distributions of sets of positive Lebesgue measure. This fact makes them natural objects to study.

We are interested in giving conditions under which certain two-dimensional maps $F$ which piecewise coincide with hyperbolic diffeomorphisms $f_i$ have SRB measures. As in the one-dimensional situation there is an essential difference between a finite and an infinite number of $f_i$. In the case of an infinite number of $f_i$, their derivatives grow with $i$ and relations between first and second derivatives become crucial.

## 2. Hyperbolicity and geometric conditions

Consider a countable collection $\xi = \{E_1, E_2, \dots,\}$ of full height closed curvilinear rectangles in $Q$. Assume that each $E_i$ lies inside a domain of definition of a $C^2$ diffeomorphism $f_i$ which maps $E_i$ onto its image $S_i \subset Q$. We assume each $E_i$ connects the top and the bottom of $Q$. Thus each $E_i$ is bounded from above and from below by two subintervals of the line segments

$$\{(x,y) : y = 1,\ 0 \le x \le 1\} \quad \text{and} \quad \{(x,y) : y = 0,\ 0 \le x \le 1\}.$$

We assume that the left and right boundaries of $E_i$ are graphs of smooth functions $x^{(i)}(y)$ with $|dx^{(i)}/dy| \le \alpha$ where $\alpha$ is a real number satisfying $0 < \alpha < 1$. We further assume that the images $f_i(E_i) = S_i$ are narrow strips connecting the left and right sides of $Q$ and that they are bounded on the left and right by the two subintervals of the line segments

$$\{(x,y) : x = 0,\ 0 \le y \le 1\} \quad \text{and} \quad \{(x,y) : x = 1,\ 0 \le y \le 1\}$$

and above and below by the graphs of smooth functions $Y^i(X)$, $\left|dY^{(i)}/dX\right| \le \alpha$. We will see later that the upper bounds on derivatives $\left|dx^{(i)}/dy\right| \le \alpha$ and $\left|dY^{(i)}/dX\right| \le \alpha$ follow from hyperbolicity conditions that we formulate below.

We call the $E_i'$s *posts*, the $S_i'$s *strips*, and we say the $E_i'$s are *full height* in $Q$ while the $S_i'$s are *full width* in $Q$.

For $z \in Q$, let $\ell_z$ be the horizontal line through $z$. We define

$$\delta_z(E_i) = \operatorname{diam}(\ell_z \cap E_i), \quad \delta_{i,\max} = \max_{z\in Q} \delta_z(E_i), \quad \delta_{i,\min} = \min_{z\in Q} \delta_z(E_i).$$

We assume the following *geometric conditions*

G1. $\operatorname{int} E_i \cap \operatorname{int} E_j = \varnothing$ for $i \ne j$.
G2. $\operatorname{mes}(Q \setminus \cup_i \operatorname{int} E_i) = 0$ where mes stands for Lebesgue measure,
G3. $-\sum_i \delta_{i,\max} \log \delta_{i,\min} < \infty$.

We emphasize that the strips $S_i$ can intersect in an arbitrary fashion, differently from condition G3 in ([**8**]).

In the standard coordinate system for a map $F : (x,y) \to (F_1(x,y), F_2(x,y))$ we use $DF(x,y)$ to denote the differential of $F$ at some point $(x,y)$ and $F_{jx}$, $F_{jy}$, $F_{jxx}$, $F_{jxy}$, etc., for partial derivatives of $F_j$, $j = 1,2$ .

Let $J_F(z) = |F_{1x}(z)F_{2y}(z) - F_{1y}(z)F_{2x}(z)|$ be the absolute value of the Jacobian determinant of $F$ at $z$.

*Hyperbolicity conditions.* — There exist constants $0 < \alpha < 1$ and $K_0 > 1$ such that for each $i$ the map

$$F(z) = f_i(z) \quad \text{for } z \in E_i$$

satisfies

H1. $\left|F_{2x}(z)\right| + \alpha\left|F_{2y}(z)\right| + \alpha^2\left|F_{1y}(z)\right| \le \alpha\left|F_{1x}(z)\right|$
H2. $\left|F_{1x}(z)\right| - \alpha\left|F_{1y}(z)\right| \ge K_0$.
H3. $\left|F_{1y}(z)\right| + \alpha\left|F_{2y}(z)\right| + \alpha^2\left|F_{2x}(z)\right| \le \alpha\left|F_{1x}(z)\right|$
H4. $\left|F_{1x}(z)\right| - \alpha\left|F_{2x}(z)\right| \ge J_F(z)K_0$.

For a real number $0 < \alpha < 1$, we define the cones

$$\begin{aligned} K_\alpha^u &= \{(v_1,v_2) : \left|v_2\right| \le \alpha\left|v_1\right|\} \\ K_\alpha^s &= \{(v_1,v_2) : \left|v_1\right| \le \alpha\left|v_2\right|\} \end{aligned}$$

and the corresponding cone fields $K_\alpha^u(z), K_\alpha^s(z)$ in the tangent spaces at points $z \in \mathbf{R}^2$.

Unless otherwise stated, we use the max norm on $\mathbf{R}^2$, $\left|(v_1,v_2)\right| = \max(\left|v_1\right|, \left|v_2\right|)$.

The following simple proposition relates conditions H1-H4 above with the usual definition of hyperbolicity in terms of cone conditions. It shows that conditions H1 and H2 imply that the $K^u_\alpha$ cone is mapped into itself by $DF$ and expanded by a factor no smaller than $K_0$ while H3 and H4 imply that the $K^s_\alpha$ cone is mapped into itself by $DF^{-1}$ and expanded by a factor no smaller than $K_0$.

***Proposition 2.1.*** — *Under conditions H1-H4 above, we have*

$$DF(K^u_\alpha) \subseteq K^u_\alpha \tag{2}$$

$$v \in K^u_\alpha \Rightarrow |DFv| \geq K_0|v| \tag{3}$$

$$DF^{-1}(K^s_\alpha) \subseteq K^s_\alpha \tag{4}$$

$$v \in K^s_\alpha \Rightarrow |DF^{-1}v| \geq K_0|v| \tag{5}$$

*Proof.* — H1 implies (2):

Let $v = (v_1, v_2) \in K^u_\alpha$. Then, $|v| = |v_1|$ since $\alpha < 1$ and $|v_2| \leq \alpha|v_1|$.

Write $DF(v_1, v_2) = (F_{1x}v_1 + F_{1y}v_2, F_{2x}v_1 + F_{2y}v_2) = (u_1, u_2)$.

Then, using H1, we have

$$\begin{aligned}
|u_2| &= |F_{2x}v_1 + F_{2y}v_2| \\
&\leq |F_{2x}||v_1| + |F_{2y}|\alpha|v_1| \\
&\leq |v_1|(|F_{2x}| + |F_{2y}|\alpha) \\
&\leq |v_1|(\alpha|F_{1x}| - |F_{1y}|\alpha^2) \\
&\leq \alpha|F_{1x}v_1 + F_{1y}v_2| \\
&= \alpha|u_1|
\end{aligned}$$

proving (2).

H2 implies (3):

Now, let $v = (v_1, v_2)$ be a unit vector in $K^u_\alpha$, so that $|v| = |v_1| = 1$ and $|v_2| \leq \alpha$.

Using H2 and the fact that $DF(v) \in K^u_\alpha$, we have

$$\begin{aligned}
|DF(v)| &= |u_1| \\
&= |F_{1x}v_1 + F_{1y}v_2| \\
&\geq |F_{1x}| - \alpha|F_{1y}| \\
&\geq K_0
\end{aligned}$$

which is (3).

The proofs that H3 and H4 imply (4) and (5) are similar using the fact that

$$DF^{-1} = \frac{1}{J_z}\begin{pmatrix} F_{2y} & -F_{1y} \\ -F_{2x} & F_{1x} \end{pmatrix}$$

This completes our proof of Proposition 2.1. □

***Remark.*** — In ([**8**]) different hyperolicity conditions were assumed which implied the invariance of cones and uniform expansion with respect to the sum norm $|v| = |v_1| + |v_2|$ (see [**3**] and [**7**] for related hyperbolicity conditions). The methods here can be adapted to work under the assumptions of ([**8**]).