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MARKOV EXTENSIONS AND DECAY OF CORRELATIONS FOR CERTAIN HÉNON MAPS

by

Michael Benedicks & Lai-Sang Young

Abstract. — Hénon maps for which the analysis in [BC2] applies are considered. Sets with good hyperbolic properties and nice return structures are constructed and their return time functions are shown to have exponentially decaying tails. This sets the stage for applying the results in [Y]. Statistical properties such as exponential decay of correlations and central limit theorem are proved.

0. Introduction and statements of results

Let $T_{a,b} : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$T_{a,b}(x, y) = (1 - ax^2 + y, bx).$$

In **[BC2]**, Carleson and the first named author developed a machinery for analyzing the dynamics of $T_{a,b}$ for a positive measure set of parameters (a, b) with $a < 2$ and b small. For lack of a better word let us call these the "good" parameters. The machinery of **[BC2]** is used in **[BY]** to prove that for every "good" pair (a, b), $T = T_{a,b}$ admits a Sinai-Ruelle-Bowen measure ν. The significance of ν is that it describes the asymptotic orbit distribution for a positive Lebesgue measure set of points in the phase space, including most of the points in the vicinity of the attractor. The aim of the present paper is to show that (T, ν) has a natural "Markov extension" with an exponentially decaying "tail", and to obtain via this extension some results on stochastic processes of the form $\{\varphi \circ T^n\}_{n=0,1,2,\ldots}$, where $\varphi : \mathbb{R}^2 \mapsto \mathbb{R}$ is a Hölder continuous random variable on the probability space $(\mathbb{R}^2, \nu)$.

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Consider in general a map $f : M \circlearrowleft$ preserving a probability measure ν. By a *Markov extension* of (f,ν) we refer to a dynamical system $F : (\Delta,\tilde{\nu}) \circlearrowleft$ and a projection map $\pi : \Delta \mapsto M$; F is assumed to have a Markov partition (with possibly infinitely many states), F and π satisfy $\pi \circ F = f \circ \pi$, and $\pi_* \tilde{\nu} = \nu$. We do not require that π be 1-1 or onto.

Let (f,ν) be as in the last paragraph, and let X be a class of functions on M. We say that (f,ν) has *exponential decay of correlations* for functions in X if there is a number $\tau < 1$ such that for every pair $\varphi, \psi \in X$, there is a constant $C = C(\varphi,\psi)$ such that

$$\left| \int \varphi(\psi \circ f^n) d\nu - \int \varphi d\nu \int \psi d\nu \right| \leq C\tau^n \qquad \forall n \geq 0.$$

Also, we say that (f,ν) has a *central limit theorem* for φ with $\int \varphi d\nu = 0$ if the stochastic process $\varphi, \varphi \circ f, \varphi \circ f^2, \ldots$ satisfies the central limit theorem, *i.e.* if

$$\frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \varphi \circ f^i \xrightarrow{\text{dist}} \mathcal{N}(0,\sigma)$$

for some $\sigma \geq 0$. For $\sigma > 0$ this means that $\forall t \in \mathbb{R}$,

$$\nu \left\{ \frac{1}{\sqrt{n}} \sum_{0}^{n-1} \varphi \circ f^i < t \right\} \longrightarrow \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{t} e^{-u^2/2\sigma^2} du$$

as $n \to \infty$.

For $f = T_{a,b}$, (a,b) "good" parameters, we have the following results:

Theorem 1 **([BY])**. — *f admits an SRB measure ν.* (See Section 1.7 for the precise definition.)

Theorem 2 **([BY])**. — *ν is the unique SRB measure for f^n for every $n \geq 1$. This implies in particular that (f^n,ν) is ergodic $\forall n \geq 1$.*

By the general theory of SRB measures, the ergodicity of (f^n,ν) for all $n \geq 1$ is equivalent to (f,ν) having the mixing property, or that it is measure-theoretically isomorphic to a Bernoulli shift, see **[L]**.

For $\gamma > 0$, let $\mathcal{H}_\gamma$ be the space of Hölder continuous functions on $\mathbb{R}^2$ with Hölder exponent γ.

Theorem 3. — *(f,ν) has exponential decay of correlations for functions in $\mathcal{H}_\gamma$. The rate of decay, τ, may depend on γ.*

Theorem 4. — *(f,ν) has a central limit theorem for all $\varphi \in \mathcal{H}_\gamma$ with $\int \varphi d\nu = 0$; the standard deviation $\sigma > 0$ iff $\varphi \neq \psi \circ f - \psi$ for some $\psi \in L^2(\nu)$.*

Theorems 1 and 2 are proved in **[BY]**, while theorems 3 and 4 are new and are proved in this paper. But since an SRB measure is constructed in the process of proving Theorem 3, this paper also contains an independent proof of Theorem 1.

Questions of ergodicity or uniqueness of SRB measures, however, are of a different nature. We will *assume* Theorem 2 for purposes of the present paper.

As mentioned earlier on, our proof of theorems 3 and 4 are carried out using a Markov extension with certain special properties. The second named author has since extended this scheme of proof to a wider setting. We will refer to **[Y]** for certain facts not specific to the Hénon maps, but will otherwise keep the discussion here as self-contained as possible.

The following is a comprehensive summary of what is in this paper, section by section.

In Section 1 we recall from **[BC2]** and **[BY]** some pertinent facts about f.

The aim of Section 2 is to clean up the notion of distance to the "critical set" previously used in **[BC2]** and **[BY]**. We prove that the various distances used before are equivalent.

Section 3 is devoted to organizing the dynamics of f in a coherent fashion. We focus on a naturally defined Cantor set Λ with a product structure defined by local stable and unstable curves and with Λ intersecting each local unstable curve in a positive Lebesgue measure set. The dynamics on Λ is analogous to that of Smale's horseshoe, except that there are infinitely many branches with variable return times. A precise description of Λ is given in Propostion A in Section 3.1.

In Section 4 we study the return time function $R : \Lambda \to \mathbb{Z}^+$, *i.e.* $z \in \Lambda$ returns to Λ after $R(z)$ iterates in the representation above. (Note that $R(z)$ is not necessarily the first return time.) We prove that the measure of $\{R > n\}$ decays exponentially fast as $n \to \infty$. This estimate is stated in Lemma 5 in Section 4.1; it plays a crucial role in the subsequent analysis.

In Section 5 we consider the quotient space $\bar{\Lambda}$ obtained by collapsing Λ along $W^s_{\rm loc}$-curves. We prove, modifying standard arguments for Axiom A systems where necessary, that $\bar{\Lambda}$ has a well defined metric structure and that the Jacobians of the induced quotient maps have a "Hölder"-type property. This step paves the way for the introduction of a Perron-Frobenius operator. The results are stated in Proposition B in Section 5.1.

Let $\overline{f^R} : \bar{\Lambda} \circlearrowleft$ denote the return map to Λ. In Section 6 we construct a tower map $F : \Delta \circlearrowleft$ over $f^R : \Lambda \circlearrowleft$ with height R (see Section 6.1). F is clearly an extension of f. A Perron-Frobenius operator is introduced for $\overline{F} : \overline{\Delta} \circlearrowleft$, the object obtained by collapsing $W^s_{\rm loc}$-curves in Δ. At this point we appeal to a theorem in **[Y]** on the spectral properties of certain abstractly defined Perron-Frobenius operators. We explain briefly how a gap in the spectrum of this operator implies exponential decay of correlation for f, referring again to **[Y]** for the formal manipulations, and finish with a proof of the Central Limit Theorem.

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1. Dynamics of certain Hénon maps

The purpose of this section is to review some of the basic ideas in [**BC2**] and [**BY**], and to set some notations at the same time. We would like to make the main ideas of this paper accessible to readers without a thorough knowledge of [**BC2**] and [**BY**], but will refer to these papers for technical information as needed. The summary in Section 1 of [**BY**] may be helpful.

1.1. General description of attractors. — In this paper we are interested in the parameter range $a < 2$ and near 2, $b > 0$ and small. The facts in Section 1.1 are elementary and hold for $f = T_{a,b}$ for an open set of parameters (a, b).

There is a fixed point located at approximately $(\frac{1}{2}, \frac{1}{2}b)$; it is hyperbolic and its unstable manifold, which we will call W, lies in a bounded region of $\mathbb{R}^2$. Let Ω be the closure of W. Then Ω is an attractor in the sense that there is an open neighborhood U of Ω with the property that $\forall\, z \in U$, $f^n z \to \Omega$ as $n \to \infty$.

Away from the y-axis, f has some hyperbolic properties. For example, let $\delta \gg b$ and let $s(v)$ denote the slope of a vector v. Then

(i) on $\{|x| \geq \delta\}$, Df preserves the cones $\{|s(v)| \leq \delta\}$;
(ii) $\exists\, M_0 \in \mathbb{Z}^+$ and $c_0 > 0$ such that if $z, fz, \ldots, f^{M-1}z \in \{|x| \geq \delta\}$ and $M \geq M_0$, then
$$\left|Df_z^M v\right| \geq e^{c_0 M}|v| \quad \forall\, v \text{ with } |s(v)| \leq \delta.$$

It is easy to show, however, that Ω is not an Axiom A attractor.

In contrast to Section 1.1, the statements in Section 1.2–1.6 hold only for a positive measure set of parameters. For the rest of this paper we fix a pair of "good" parameters (a, b) and write $f = T_{a,b}$.

1.2. The critical set. — A subset $\mathcal{C} \subset W$, called the *critical set*, is designated to play the role of critical points for 1-dimensional maps. Points in $\mathcal{C}$ have x-coordinates ≈ 0; they lie on $C^2(b)$ segments of W (a curve is called $C^2(b)$ if it is the graph of a function $y = \varphi(x)$ with $|\varphi'|$, $|\varphi''| \leq 10b$); and they have "homoclinic" behaviour in the sense that if τ denotes a unit tangent vector to W, then for $z \in \mathcal{C}$, $\left|Df_z^j \tau\right| \leq (5b)^j\ \forall\, j \geq 0$.

Other important properties of $z \in \mathcal{C}$ are that $\forall\, n \geq 1$:

(i) $\left|Df_z^n \binom{0}{1}\right| \geq e^{c(n-1)}$ for some $c \approx \log 2$;
(ii) "dist"$(f^n z, \mathcal{C}) > e^{-\alpha n}$ for some small $\alpha > 0$. (The precise meaning of "dist" will be given shortly.)