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TOTAL DISCONNECTEDNESS OF JULIA SETS AND ABSENCE OF INVARIANT LINEFIELDS FOR REAL POLYNOMIALS

by

Genadi Levin & Sebastian van Strien

Abstract. — In this paper we shall consider real polynomials with one (possibly degenerate) non-escaping critical (folding) point. Necessary and sufficient conditions are given for the total disconnectedness of the Julia set of such polynomials. Also we prove that the Julia sets of such polynomials do not carry invariant linefields. In the real case, this generalises the results by Branner and Hubbard for cubic polynomials and by McMullen on absence of invariant linefields.

1. Introduction

In a paper by Branner and Hubbard [**BH**], cubic polynomials were considered, and the problem was solved when the Julia set of such a polynomial is totally disconnected (for the history of this problem see [**BH**], Ch. 5). In the same paper, the question was raised whether this result could be extended to polynomials of higher degrees. The method and results of [**BH**] hold for polynomials P of higher degrees with all but one critical points escaping to infinity, under the condition that the unique non-escaping critical point c is *simple*: $P'(c) = 0$, $P''(c) \neq 0$, see [**BH**], Ch. 12.

On the other hand, if the non-escaping critical point c is *multiple*, *i.e.*,

$$P''(c) = \cdots = P^{(\ell-1)}(c) = 0, \quad P^{(\ell)}(c) \neq 0,$$

for some $\ell > 2$, the method of [**BH**] breaks down (see [**Dou1**] for a discussion on this). The positive integer ℓ is called the *multiplicity*, or *local degree* of the critical point c of the polynomial P.

In this paper we shall prove

Theorem 1.1. — *Let P be a polynomial with real coefficients, such that one (maybe multiple) critical point c of P of even multiplicity ℓ has a bounded orbit, and all other critical points escape to infinity. Then*

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– *the filled Julia set of* P*:*

$$K(P) = \{z : \{P^n(z)\}_{n=0}^{\infty} \text{ is bounded}\}$$

is totally disconnected if and only if the connected component of the real trace $K(P) \cap \mathbb{R}$ *of the filled Julia set containing the critical point* c*, is equal to a point*

– *the Julia set* $J(P) = \partial K(P)$ *carries no measurable invariant linefields.*

Remark 1.1. — For the case that the multiplicity is odd, see **[LS2]**.

Remark 1.2. — The map $P\colon \mathbb{R} \to \mathbb{R}$ does not have a wandering interval (on the real line) with bounded orbit **[MS]**. Hence, the condition: "the connected component of the real trace $K(P) \cap \mathbb{R}$ of the filled Julia set which contains the critical point c, is equal to a point" is equivalent to one of the following conditions:

– the component of the filled Julia set $K(P)$ containing the non-escaping critical point, is non-periodic;
– P does not have an attracting or neutral periodic orbit, and is not renormalizable on the real line (*i.e.*, there is no interval I around $c = 0$, such that $P^i(I) \cap P^j(I) = \varnothing$ for $0 \le i < j \le q-1$ and $P^q(I) \subset I$);
– the intersections of the critical puzzle pieces with the real line shrink to the point c (see the next Section).

Remark 1.3. — We allow escaping critical points to be non real. Of course, since P is real, the orbit of the unique non-escaping critical point is real. The theorem holds in particular for maps of the form $f(z) = z^{\ell} + c_1$ when c_1 is real.

The second part of the Theorem extends the main result of McMullen in **[McM]**. As usual, we say that the Julia set $J(P)$ of P carries a *measurable invariant line field* if there exists a measurable subset E of the Julia set of P and a measurable map which associates to Lebesgue almost every $x \in E$ a line $l(x)$ through x which is P-invariant in the sense that $l(P(x)) = DP(x)\, l(x)$. (So the absence of linefields is obvious if the Julia set has zero Lebesgue measure.) The absence of invariant linefields was proved by McMullen for all maps of the form $P(z) = z^{\ell} + c_1$, ℓ is even and c_1 is real, which are infinitely renormalizable. If P is quadratic (*i.e.*, $\ell = 2$) and only finitely often renormalizable then this holds because then the corresponding parameter c_1 lies at the boundary of the Mandelbrot set, see **[Y]**, **[H]**. (Actually, the result of Yoccoz is much stronger: local connectivity of the boundary of the Mandelbrot set at such points).

The (non-)existence of the invariant linefields is strongly related to the Density of Hyperbolicity Conjecture, see **[MSS]**. It follows from the second part of the Theorem, because of Theorem E of **[MSS]**, that for any polynomial P as in the Theorem, there exists another (maybe complex) polynomial Q of the same degree and with the same multiplicity ℓ of the non-escaping critical point $c = 0$, which is as close to P as we wish and such that Q is hyperbolic: every critical point of Q tends either to infinity, or to an attracting periodic orbit. (One can assume that $P(z) = z^\ell \cdot p(z) + t$, where $p(z) = z^m + \cdots + p_m$ is a monic polynomial of the degree $m \geq 0$, and $p_m \neq 0$. Then the polynomial Q as above is of the same form, and P and Q are considered as points of the space $\mathbb{C}^m \times \mathbb{C}$). In fact, for $\ell = 2$ a much stronger statement is true, since the density of hyperbolicity within real quadratic maps implies that one chooses Q as above real.

The proof of the Theorem is postponed until Section 3, and is based on Propositions 2.1- 2.4 of the next section.

Propositions 2.1 and 2.3 give sufficient conditions for the total disconnectedness of Julia set and absence of invariant linefields. They could be applied to complex polynomials as well. Nevertheless, we can only show that this condition is satisfied in the case considered in the theorem: see Proposition 2.4 and Section 3.

A similar problem exists for odd multiplicities. If P is a polynomial as in the Theorem, but with ℓ an odd number (say, cubic one), then the Theorem is easy if there are no other critical points (*i.e.*, $P(z) = z^\ell + c_1$, where c_1 is real and ℓ is odd), because then the map is monotone on the real line. On the other hand, if other (escaping) critical points exist, we can still apply Propositions 2.1, but the implementation of it (a statement like Proposition 2.4) uses different methods, see **[Le]** and **[LS2]**.

Before giving the proofs, let us make a remark about the non-minimal case (*i.e.*, when the postcritical set $\omega(c)$ contains c, but the system restricted to $\omega(c)$ is not minimal). Such system is relatively simple when it is *real* (see Proposition 3.2 of **[LS1]** and Proposition 2.5 below, or see **[Ly]**). But this is definitely not the case for complex parameters:

Remark 1.4. — In each of Douady's examples of an infinitely renormalizable quadratic map f with a non-locally connected Julia set, the postcritical set is non-minimal. Indeed, according **[P-M]** there exists an invariant Cantor set on which the map is injective. If this Cantor set does not intersect the postcritical set, then according to a well-known result of Mañé **[Ma]** the map is expanding on this set. This is impossible since f is injective on this set, see **[Dou2]**.

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2. Associated mappings and complex bounds

Let $G\colon \cup_{i=0}^{i_0} \Omega^i \to \Omega$ be an *ℓ-polynomial-like mapping*. As in **[DH]**, **[LM]**, **[LS1]** this means that all Ω^i are open topological discs with pairwise disjoint closures which are compactly contained in the topological disc Ω, the map $G\colon \Omega^0 \to \Omega$ is ℓ-to-one holomorphic covering with a unique critical point $c = 0 \in \Omega^0$, and that each other map $G\colon \Omega^i \to \Omega$ is a conformal isomorphism. Its *filled Julia set* is defined by

$$K(G) = \{z : G^n(z) \in \cup_{i=0}^{i_0} \Omega^i, n = 0, 1, \dots\}.$$

The boundary $\partial K(G)$ is called the *Julia set* $J(G)$ of G.

The *puzzle* (see **[BH]**) of the map G is said to be the set of connected components of all preimages $G^{-k}(\Omega)$, $k = 0, 1, \dots$. A *piece* of level $k \geq 0$ is a connected component of $G^{-k}(\Omega)$. A piece is *critical* if it contains the critical point c of G. We call another ℓ-polynomial-like mapping $G'\colon \cup_{i=0}^{i'_0} \Omega'^i \to \Omega'$ *associated to* or *induced by* G if G' restricted to each Ω'^i is some iterate $G^{j(i)}$ of G. We also call G *real* iff all topological discs Ω^i, Ω are symmetric w.r.t. the real axis, and $\overline{G(z)} = G(\overline{z})$, for any $z \in \cup_{i=0}^{i_0} \Omega^i$. In particular, this implies that the postcritical set of the unique critical point $c = 0 \in \Omega_0$ is real.

Proposition 2.1. — *Fix a real ℓ-polynomial-like mapping G. Assume there exists an infinite sequence $G(j)\colon \cup_i \Omega^i(j) \to \Omega(j)$ of real ℓ-polynomial-like mappings associated to the mapping G with $\omega(c)$ minimal such that the critical point $c = 0 \in \Omega^0(j)$ of $G(j)$ does not escape the domain of $G(j)$ under iterations of $G(j)$. Assume moreover that*

(1) *each $\Omega^i(j) \cap \mathbb{R}$ coincides with the intersection of some piece of G with the real line;*

(2) *when $G^i(c) \in \Omega^0(j)$ then $G^i(c)$ is an iterate of c under $G(j)$ (we call this* the first return condition for G on $\Omega^0(j)$*);*

(3) *the modulus of the annuli $\Omega(j) \setminus \Omega^0(j)$ is uniformly bounded away from zero by a constant $m > 0$ which does not depend on j;*

(4) *the diameter of $\Omega(j)$ tends to zero as $j \to \infty$.*

Then the filled Julia set of G is totally disconnected.

Remark 2.1. — Conditions (1) and (4) obviously imply the third condition of Remark 1.2: the traces of the critical pieces on the real line shrink to the point.

Remark 2.2. — The proposition also holds for a complex (*i.e.*, not real) map G, if one replaces (1) by the following condition:

for each j, one can find another ℓ-polynomial-like mapping $R(j)$ having as its range a critical piece $P(j)$ of G, so that the two mappings $G(j)$ and $R(j)$ satisfy the conditions of Proposition 2.1 from **[LS1]**.

Proof. — Let us first observe that the first return condition also holds for the first return map of $G(j)$ to $\Omega^0(j)$. Indeed, consider the first return map of $G(j)$ to $\Omega^0(j)$ along the iterates of the critical point. This is again a real ℓ-polynomial-like mapping $\tilde{G}(j)\colon \cup_i \tilde{\Omega}^i(j) \to \Omega^0(j)$. Obviously, if $G^i(c) \in \Omega^0(j)$ then $G^i(c)$ is an iterate of c under $\tilde{G}(j)$. In addition, the modulus of $\Omega^0(j) \setminus \tilde{\Omega}^0(j)$ is greater or equal to $1/\ell$ mod $(\Omega(j) \setminus \Omega^0(j))$. So we may replace $G(j)$ by the first return map to $\Omega^0(j)$ and therefore in the remainder of the proof we can and will assume that the first return