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POSITIVE LYAPUNOV EXPONENTS FOR LORENZ-LIKE FAMILIES WITH CRITICALITIES

by

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Dedicated to Adrien Douady on the occasion of his 60^{th} *birthday*

Abstract. — We introduce a class of one-parameter families of real maps extending the classical geometric Lorenz models. These families combine singular dynamics (discontinuities with infinite derivative) with critical dynamics (critical points) and are based on the behaviour displayed by Lorenz flows over a fairly wide range of parameters. Our main result states that – nonuniform – expansion is the prevalent form of dynamics even after the formation of the criticalities.

1. Introduction and statement of results

Numerical analysis of the now famous system of differential equations

$$(1) \qquad \begin{cases} \dot{x} = -\sigma x + \sigma y \\ \dot{y} = rx - y - xz \\ \dot{z} = -bz + xy \end{cases}$$

for parameter values $r \approx 28$, $\sigma \approx 10$, $b \approx 8/3$, led Lorenz **[11]** to identify sensitive dependence of orbits with respect to the corresponding initial points as a main source of unpredictability in deterministic dynamical systems. His observations were then interpreted by **[1]**, **[6]**, who described expanding ("strange") attractors in certain geometric models for the behaviour of (1). Conjecturedly, such an attractor exists also for Lorenz' original equations, although this has not yet been proved.

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Further study of (1) revealed that relatively small variations of these parameter values may lead to quite different, albeit even more complex, dynamical features. Indeed, already as r is increased past $r \approx 30$ Poincaré return maps cease to be described by the cusp-type pictures corresponding to the geometric models, instead they exhibit "folded cusps", or "hooks"; moreover, these hooks persist in a large window of values of r (extending beyond $r \approx 50$), see **[15]** for a thorough discussion. Trying to understand this folding process and its effect on the behaviour of the flow was, in fact, a main motivation behind Hénon's model of strange attractor for maps in two dimensions, **[7]**, **[8]**. In constructing this model he focused on the dynamics near the fold, in particular disregarding trajectories which pass close to equilibrium points.

Here we aim at a more global understanding of the dynamics of Lorenz flows, accounting for the interaction between *singular behaviour* (corresponding to trajectories near equilibria) and *critical behaviour* (near folding regions). Indeed, we introduce a one-dimensional prototype for this problem, largely inspired by the observations in **[15]**, which we call Lorenz-like families with criticalities. Apart from their present motivation, these families of maps are also of interest in their own right, as models of rich nonsmooth dynamics in dimension one. Moreover, in an ongoing work we are further pushing the present constructions and conclusions to the context of smooth flows in three-dimensional space, *cf.* comments below.

Let us begin by explaining what we mean by *Lorenz-like families with criticalities.* We consider one-parameter families $\{\varphi_a\}$ of real maps of the form

$$\varphi_a(x) = \begin{cases} \varphi(x) - a & \text{if } x > 0 \\ -\varphi(-x) + a & \text{if } x < 0 \end{cases}$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is smooth and satisfies:

L1 : $\varphi(x) = \psi(x^\lambda)$ for all $x > 0$, where $0 < \lambda < 1/2$ and ψ is a smooth map defined on $\mathbb{R}$ with $\psi(0) = 0$ and $\psi'(0) \neq 0$;

L2 : there exists some $c > 0$ such that $\varphi'(c) = 0$;

L3 : $\varphi''(x) < 0$ for all $x > 0$.

As we already mentioned, this definition is motivated by a fair amount of numerical and analytical data concerning the behaviour of Lorenz flows. In particular, the condition $\lambda < 1/2$ corresponds to the fact that, for the parameter region we are interested in, the expanding eigenvalue λ_u of (1) at the origin is more than twice stronger than the weakest contracting eigenvalue λ_s (that is $\lambda_u + 2\lambda_s > 0$).

For small values of the parameter the maximal invariant set of φ_a in the interval $[-a, a]$ is a hyperbolic Cantor set. Under certain natural conditions, implied by L4 and L5 below, the entire interval $[-a, a]$ becomes forward invariant as a crosses some value $a_1 > 0$. This situation persists for a certain range of parameter values and corresponds to the class of maps usually associated to the "Lorenz attractor" (see **[5]**, **[6]**, **[1]**). The dynamics of such maps is relatively well understood: they admit an

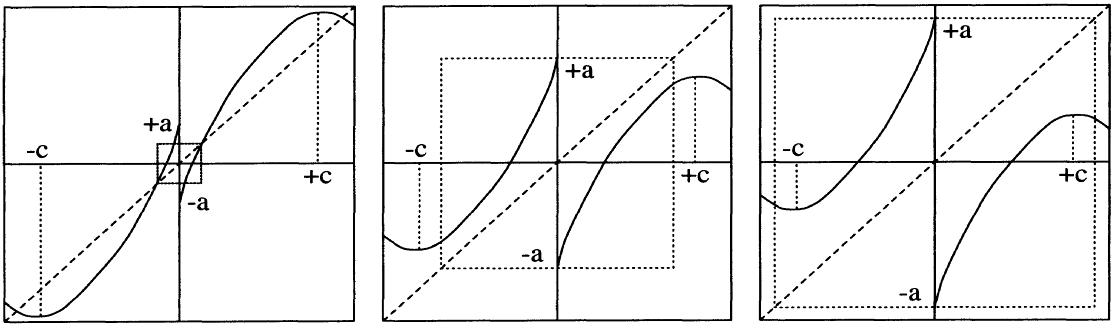

FIGURE 1. Lorenz-like families with criticalities

invariant measure which is absolutely continuous with respect to Lebesgue and has positive metric entropy; they are not structurally stable but are fully persistent in the sense that any small perturbation also admits an absolutely continuous invariant measure of positive entropy.

We are mainly interested in studying the bifurcation which occurs as the parameter crosses the value $a = c$. With this in mind, we add two natural assumptions on φ which ensure that a Lorenz attractor persists for all $a < c$.

Let $x_{\sqrt{2}}$ denote the unique point in $(0, c)$ such that $\varphi'(x_{\sqrt{2}}) = \sqrt{2}$; sometimes we also write $a_2 = x_{\sqrt{2}}$. Then we suppose

L4 : $0 < \varphi_a(x_{\sqrt{2}}) < \varphi_a(a) < x_{\sqrt{2}}$ for all $a \in (a_2, c]$.

The last inequality implies that given any y with $|y| \in [x_{\sqrt{2}}, a)$ there exists a unique $x \in [-a, a]$ such that $\varphi_a(x) = y$. Note that x and y have opposite signs. Moreover, the first inequality implies that $|x| < x_{\sqrt{2}}$. Our last assumption is

L5 : $|(\varphi_c^2)'(x)| > 2$ for all $x \in [-c, c] \setminus \{0\}$ such that $|\varphi_c(x)| \in [x_{\sqrt{2}}, c]$.

Observe that this is automatic if $\varphi_c(x) = x_{\sqrt{2}}$ (because $|x|$ is strictly smaller than $x_{\sqrt{2}}$, by the previous remarks) and also if $\varphi_c(x)$ is close to c (then x is close to zero and so $|(\varphi_c^2)'(x)| \approx |x|^{2\lambda - 1} \approx \infty$).

It is straightforward to check that L1-L5 are satisfied by a nonempty open set of one-parameter families, where openness is meant with respect to the C^2 topology in the space of real maps ψ. Moreover, we shall show that these hypotheses do imply that φ_a is essentially uniformly expanding for all parameters up to c:

Proposition 1.1. — *Given any* $a \in [a_1, c]$,

(1) *the interval* $[-a, a]$ *is forward invariant and* $\varphi | [-a, a]$ *is transitive*

(2) $|(\varphi_a^n)'(x)| \geq \min\{\sqrt{2}, |\varphi_a'(x)|\}(\sqrt{2})^{n-1}$ *for all* $x \in [-a, a]$ *such that* $\varphi_a^j(x) \neq 0$ *for every* $j = 0, 1, \ldots, n - 1$.

After the bifurcation $a = c$ such uniform expansivity is clearly impossible, due to the presence of the critical point in the domain of the map. However, our main result states that – nonuniform – expansivity persists, in a measure-theoretical sense, and is even the prevalent form of dynamics after the bifurcation. We denote $c_j(a) = \varphi_a^j(c)$ for each $j \geq 1$.

Theorem. — *Let $\{\varphi_a\}$ be a Lorenz-like family satisfying conditions L1-L5. Then there are $\sigma > 0$ and $\mathcal{A}^+ \subset \mathbb{R}$ such that $|(\varphi_a^j)'(c_1(a))| \geq e^{\sigma j}$ for all $a \in \mathcal{A}^+$ and $j \geq 1$ and*

$$\lim_{\varepsilon \to 0} \frac{m(\mathcal{A}^+ \cap [c, c+\varepsilon])}{\varepsilon} = 1 \qquad (m = \textit{Lebesgue measure on } \mathbb{R}).$$

Moreover, there is $\sigma_1 > 0$ such that if $a \in \mathcal{A}^+$ then for m-almost all $x \in [-a, a]$ we have $\limsup \frac{1}{n} |\log (\varphi_a^n)'(x)| \geq \sigma_1$ *as $n \to \infty$.*

Measure theoretic persistence of positive Lyapunov exponents (outside the class of uniformly expanding maps) was first proved by Jakobson **[9]**, for maps in the quadratic family $f_a(x) = 1 - ax^2$ close to parameter values $\bar{a}$ satisfying (**[12]**)

$$(2) \qquad \inf_{j \in \mathbb{N}} \left| f_{\bar{a}}^j(c) - c \right| > 0 \qquad (c = \text{critical point} = 0).$$

There exist today many proofs of this theorem, *e.g.* **[4]**, **[2]**, as well as generalizations to families of smooth maps with finitely many critical points **[16]**, and to families of maps in which a single discontinuity coincides with the critical point **[14]**. A number of differences should be pointed out in this setting, between smooth maps and our Lorenz-like maps.

While all proofs of Jakobson's theorem in the smooth context rely in one way or the other on the nonrecurrence condition (2), here we need no assumption on the orbits of the critical points for $a = c$. Instead, we simply take advantage of the strong expansivity estimates given by Proposition 1.1 for that parameter value.

Various technical complications arise in the present situation from the existence of discontinuities and of regions where the derivative has arbitrarily large norm. Several estimates (including distortion bounds), which in the smooth case rely on the boundedness and Lipschitz continuity of the derivative, now require nontrivial reformulations together with a detailed study of the recurrence near the discontinuity (and not only near the critical points).

Lorenz-like families with criticalities undergo codimension-one bifurcations which mark a direct transition from uniformly expanding dynamics (for $a < c$), to nonuniformly expanding dynamics (for $a \in \mathcal{A}^+$), a kind of bifurcation which does not seem to be known in the smooth one-dimensional context. The fact that the bifurcation parameter $a = c$ is a Lebesgue density point for $\mathcal{A}^+$ is related to the strong form of expansivity exhibited by φ_c. An interesting question is whether some characterization of the density points of $\mathcal{A}^+$ can be given in terms of special hyperbolicity features of the corresponding maps (*e.g.* uniformly hyperbolic structure on periodic orbits ?).

We remark here that the symmetry inherent in our definition of Lorenz-like maps, though partly justified by the symmetry which exists in Lorenz' system of equations,