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INVARIANT MEASURES FOR TYPICAL QUADRATIC MAPS

by

Marco Martens & Tomasz Nowicki

Abstract. — A sufficient geometrical condition for the existence of absolutely continuous invariant probability measures for $S-$unimodal maps will be discussed. The Lebesgue typical existence of Sinai-Bowen-Ruelle-measures in the quadratic family will be a consequence.

1. Introduction

A general belief, or hope, in the theory of dynamical systems is that typical dynamical systems have well-understood behavior. This belief has two forms, depending on the meaning of the word "typical". It could refer to the topological generic situation or to the Lebesgue typical situation in parameter space. In this work *typical* will refer to Lebesgue typical and the behavior of a Lebesgue typical quadratic map on the interval will be discussed.

The quadratic family is formed by the maps $q_t : [-1,1] \to [-1,1]$ with $t \in [0,1]$ and

$$q_t(x) = -2tx^\alpha + 2t - 1,$$

with the critical exponent $\alpha = 2$. The maps in this family can be classified as follows. The maps in

$$\mathcal{P} = \{t \in [0,1] \mid q_t \text{ has a periodic attractor}\}$$

have a unique periodic orbit whose basin of attraction is an open and dense set. Moreover this basin has full Lebesgue measure. In particular the invariant measure on the periodic attractor is the SBR-measure for the map. Recall that a measure μ on $[-1,1]$ is called an S(inai)-B(owen)-R(uelle)-measure for q_t if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \delta_{q_t^k(x)} = \mu,$$

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for typical $x \in [-1, 1]$.

The maps in

$$\mathcal{R} = \{t \in [0, 1] \mid q_t \text{ is infinitely renormalizable}\}$$

have a unique invariant minimal Cantor set which attracts both generic and typical orbits. This Cantor set is uniquely ergodic and has zero Lebesgue measure, **[BL2]**, **[G]**, **[M1]**. The unique invariant measure on this Cantor set is the SBR-measure for the system. The maps in

$$\mathcal{I} = [0, 1] \setminus \{\mathcal{P} \cup \mathcal{R}\}$$

have a periodic interval whose orbit is the limit set of generic orbits. The orbit of this periodic interval absorbs also the orbit of typical points. These maps are ergodic with respect to the Lebesgue measure, **[BL1]**, **[GJ]**, **[K]**, **[M1]**. In the quadratic family, $\alpha = 2$, the limit set of typical points is actually also the orbit of this periodic interval, **[L1]**. However, in families with α big enough there are maps in $\mathcal{I}$ whose typical limit set is not this periodic interval, **[BKNS]**.

Before discussing the behavior of typical quadratic maps let us include the behavior of generic quadratic maps .

Theorem 1.1 **([GS], [L3])**. — *Hyperbolicity is dense in the quadratic family, e.g.* $\overline{\mathcal{P}} = [0, 1]$.

We will continue to specify the behavior of a typical map in $\mathcal{I}$. The dynamics of maps in

$$\mathcal{M} = \{t \in \mathcal{I} \mid q_t \text{ has an absolutely continuous invariant probability measure}\}$$

is well-understood. The measure is unique and its support is the orbit of the above periodic interval. Moreover it has positive Lyapunov exponent, **[K]**, **[Ld]**. Starting in **[NU]**, where it was shown that $q_1 \in \mathcal{M}$, more and more maps q_t were shown to have such a measure (**[B]**, **[R]**, **[Mi]**). Finally it was shown in **[Ja]** that $\mathcal{M}$ has positive measure.

Main Theorem 1.2 **(joint with Lyubich)**. — *A typical quadratic map has a unique SBR-measure. More specifically*

(1) *for $t \in \mathcal{P}$ the support of the SBR-measure is the periodic attractor,*
(2) *for $t \in \mathcal{R}$ the SBR-measure is supported on a Cantor set,*
(3) *for $t \in \mathcal{M}$ the SBR-measure is an absolutely continuous measure supported on the orbit of a periodic interval,*
(4) *the set $\mathcal{P} \cup \mathcal{R} \cup \mathcal{M} \subset [0, 1]$ has full Lebesgue measure.*

The general belief that typical dyamical systems have well understood behavior has been precisely formulated in the Palis-Conjecture **[P]**. Now, by Theorem 1.2, this Conjecture has been proved for the quadratic family.

Johnson constructed unimodal maps in $\mathcal{I}$ (with arbitrary critical exponent) which do not have an absolutely continuous invariant measure, [**Jo**]. More careful combinatorial Johnson-Examples were made without SBR-measure [**HK**]. The same work shows the existence of maps in $\mathcal{I} \setminus \mathcal{M}$ which have an SBR-measure but this measure is not absolutely continuous. The complications which occur in $\mathcal{I} \setminus \mathcal{M}$ are thoroughly studied in [**Br**].

In this work we will formulate a geometrical condition on maps in $\mathcal{I}$ sufficient for the existence of absolutely continuous invariant probability measures. The geometric condition is formulated in terms of a decreasing sequence of *central intervals* $U_n = (-u_n, u_n)$, $n \geq 1$, which are defined for all unimodal maps with recurrent critical orbit. The domain $D_n \subset U_n$ of the first return map $R_n : D_n \to U_n$, $n \geq 1$ is a countable collection of intervals. The central component of D_n is U_{n+1}. The first return map R_n is said to have a *central return* when

$$R_n(0) \in U_{n+1}.$$

The sufficient geometrical condition for the existence of absolutely continuous measures is stated in terms of *scaling factors*

$$\sigma_n = \frac{u_{n+1}}{u_n}, \quad n \geq 1.$$

These scaling factors describe the small scale geometrical properties of the system but they are also strongly related to distortion questions. The main consequence of the distortion Theory developed in [**M1**] are the a priori bounds on the distortion of each R_n. The renormalization Theory developed in [**L1**] and [**LM**] achieved much stronger results: if a quadratic unimodal map has only finitely many central returns then the scaling factors tend exponentially to zero.

The scaling factors are related to small scale geometry, distortion but also expansion. The technical step in this work is to show that small scaling factors imply strong expansion along the critical orbit. In [**NS**] it was shown that enough expansion along the critical orbit causes the existence of an absolutely continuous invariant probability measure. In particular, if

$$\sum_{n \geq 1} \left| Dq_t^n(q_t(0)) \right|^{-1/\alpha} < \infty$$

then q_t has an absolutely continuous invariant probability measure.

Main Theorem 1.3. — *Let f be an $S-$unimodal map with critical exponent $\alpha > 1$. If f has summable scaling factors, that is*

$$\sum_{n \geq 1} \sigma_n^{1/\alpha} < \infty,$$

then it has an absolutely continuous invariant probability measure.

Corollary 1.4. — *If a quadratic map has only finitely many central returns then it has an absolutely continuous invariant probability measure.*

The (Johnson-)Examples in **[Jo]** have infinitely many (cascades of) central returns. The corollary states that the only quadratic unimodal maps in $\mathcal{I}$ which do not have an absolutely continuous invariant measure are Johnson-Examples. The families $\{q_t\}$ with α big enough have maps in $\mathcal{I}$ which do not have an absolutely continuous invariant probability measure and which are also not Johnson-Examples, **[BKNS]**.

In **[L2]**, Lyubich studies the parameter space of the (holomorphic) quadratic family. A new proof showing that $\mathcal{I}$ has positive Lebesgue measure is given (compare with the Jacobson-Theorem **[Ja]**). Moreover it is shown that for almost every parameter in $\mathcal{I}$ the corresponding quadratic map has only finitely many central returns. This, together with Theorem 1.3, implies Theorem 1.2.

Conjecture 1.5. — *A typical map in the family $\{q_t\}$, with critical exponent $\alpha > 1$, has a unique SBR-measure. More specifically*

(1) *for $t \in \mathcal{P}$ the support of the SBR-measure is the periodic attractor,*
(2) *for $t \in \mathcal{R}$ the SBR-measure is supported on a Cantor set,*
(3) *for $t \in \mathcal{M}$ the SBR-measure is an absolutely continuous measure supported on the orbit of a periodic interval,*
(4) *the set $\mathcal{P} \cup \mathcal{R} \cup \mathcal{M} \subset [0,1]$ has full Lebesgue measure.*

An appendix is added to collect the standard notions and Lemmas in interval dynamics.

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2. Central Intervals

Throughout the following sections we will fix an S−unimodal map $f : [-1,1] \to [-1,1]$ with critical exponent $\alpha > 1$ and without periodic attractors. Furthermore assume that the critical orbit is recurrent.

The set of nice points is

$$\mathcal{N} = \{x \in [-1,1] \mid \forall i \geq 0 \; f^i(x) \notin (-|x|,|x|)\}$$

This set is closed and not empty. For example the fixed point of f in $(0,1)$ is in $\mathcal{N}$.

For $x \in \mathcal{N}$ let $D_x \subset U_x = (-|x|,|x|)$ be the set of points whose orbit returns to U_x. The first return map to U_x is denoted by

$$R_x : D_x \longrightarrow U_x.$$

The next Lemma is a straightforward consequence of the fact that the boundary of each U_x is formed by nice points.