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A GLOBAL VIEW OF DYNAMICS AND A CONJECTURE ON THE DENSENESS OF FINITUDE OF ATTRACTORS

by

Jacob Palis

To Adrien Douady for his lasting contribution to mathematics (July 1995)

Abstract. — A view on dissipative dynamics, i.e. flows, diffeomorphisms, and transformations in general of a compact boundaryless manifold or the interval is presented here, including several recent results, open problems and conjectures. It culminates with a conjecture on the denseness of systems having only finitely many attractors, the attractors being sensitive to initial conditions (chaotic) or just periodic sinks and the union of their basins of attraction having total probability. Moreover, the attractors should be stochastically stable in their basins of attraction. This formulation, dating from early 1995, sets the scenario for the understanding of most nearby systems in parametrized form. It can be considered as a probabilistic version of the once considered possible existence of an open and dense subset of systems with dynamically stable structures, a dream of the sixties that evaporated by the end of that decade. The collapse of such a previous conjecture excluded the case of one dimensional dynamics: it is true at least for real quadratic maps of the interval as shown independently by Swiatek, with the help of Graczyk [GS], and Lyubich [Ly1] a few years ago. Recently, Kozlovski [Ko] announced the same result for C^3 unimodal mappings, in a meeting at IMPA. Actually, for one-dimensional real or complex dynamics, our main conjecture goes even further: for most values of parameters, the corresponding dynamical system displays finitely many attractors which are periodic sinks or carry an absolutely continuous invariant probability measure. Remarkably, Lyubich [Ly2] has just proved this for the family of real quadratic maps of the interval, with the help of Martens and Nowicki [MN].

1. Introduction and Main Conjecture

In the sixties two main theories in dynamics were developed, one of which was designed for conservative systems and called KAM for Kolmogorov-Arnold-Moser. A later important development in this area, in the eighties, was the Aubry-Mather theory

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for periodic (and Cantori) motions, which has been more recently further improved by Mather. Other outstanding results have been obtained even more recently by Eliasson, Herman, Mañé and others.

The focus of this paper, however, will be the surprising unfolding of the other theory that has been constructed for general systems (nonconservative, dissipative) and called hyperbolic: it deals with systems with hyperbolic limit sets. This means for a diffeomorphism f on a manifold M, that the tangent bundle to M at L, the limit set of f, splits up into df-invariant continuous subbundles $T_L M = E^s \oplus E^u$ such that $df \mid E^s$, $df^{-1} \mid E^u$ are contractions, with respect to some Riemmanian metric. As for most of the concepts in the sequel, a similar definition holds for a non-invertible map g, requiring $dg \mid E^u$ to be invertible. And for a flow X_t, $t \in \mathbb{R}$, we add to the splitting a subbundle in the direction of the vector field that generates the flow

$$T_L M = E^s \oplus E^u \oplus E^0$$

and require for some Riemannian metric and some constants C, $0 < \lambda < 1$, that

$$||dX_t \mid E^s||, \quad ||dX_{-t} \mid E^u|| \leq C\, e^{\lambda t}, \quad t \in \mathbb{R}.$$

See **[PT]**, specially chapter seven, for details and many of the notions presented here.

The concept of hyperbolicity was introduced by Smale, with important contributions to its development as a theory being also given by some of his students at the time, as well as Anosov, Arnold, Sinai and others. Initially, it was created to help pursue the "lost dream" referred to in the abstract: to find an open and dense subset of dynamically (structurally) stable systems; i.e., systems that when slightly perturbed in the C^r-topology, $r \geq 1$, remain with the same dynamics, modulo homeomorphisms of the ambiente space that preserve orbit structures, in the case of flows, or are conjugacies, in the case of transformations. It has actually transcended this objective, loosing through a series of counter-examples its projected character of much universality, i.e. its validity for an open dense subset of systems. But it became the ground basis for a notable evolution that dynamics experienced in the last twenty five years or so. Still, based on previous results, specially by Anosov and by ourselves, Smale and I were able to formulate in 1967 what was called the Stability Conjecture, that would fundamentally tie together hyperbolicity and dynamical stability: a system is C^r-stable if its limit set is hyperbolic and, moreover, stable and unstable manifolds meet transversally at all points. For stability restricted to the limit set, the transversality condition is substituted by the nonexistence of cycles among the transitive (dense orbit), hyperbolic subsets of the limit set.

The theory of hyperbolic systems, i.e. systems with hyperbolic limit sets, was quite developed especially for flows and diffeomorphisms, and it was perhaps even near completion (an exageration!), by the end of the sixties and beginning of the seventies. That included some partial classifications, and an increasing knowledge of their ergodic properties, due to Sinai, Bowen, Ruelle, Anosov, Katok, Pesin, Franks, Williams, Shub, Manning, among several others.

More or less at the same time, the proof of one side of the Stability Conjecture was completed through the work of Robbin, Robinson and de Melo. However, the outstanding part of this basic question from the 60's was proved to be true only in the middle 80's, in a remarkable work of Mañé **[M2]** for diffeomorphisms, followed ten years later by an again remarkable paper of Hayashi **[Ha]** for flows: C^1 dynamically stable systems must be hyperbolic. Before, by 1980, Mañé had proved the two-dimensional version of the result, but independent and simultaneous proofs were also provided by Liao and Sannami. Other partial contributions should be credited to Pliss, Doering, Hu and Wen. A high point in Hayashi's work is his connecting lemma creating homoclinic orbits by C^1 small perturbations of a flow or diffeomorphism: an unstable manifold accumulating on some stable one can be C^1 perturbed to make it intersect one another (the creation of homoclinic or heteroclinic orbits). This fact has been at this very moment sparkling some advance of dynamics in the lines proposed here, as it will be pointed out later.

While the ergodic theory of dynamical systems, as suggested by Kolmogorov and more concretely by Sinai, was being successfully developed, the hope of proposing some global structure for dynamics in general grew dimmer and dimmer in the seventies. This was due to new intricate dynamical phenomena that were presented or suggested all along the decade. First, Newhouse **[N]** extended considerably his previous results, showing that infinitely many sinks occur for a residual subset of an open set of C^2 surface diffeomorphisms near one exhibiting a homoclinic tangency. Perhaps equally or even more striking at the time, was the appearance of attractors having sensitivity with respect to initial conditions in their basin.

Although there are several possible definitions of an attractor, here we will just require it to be invariant, transitive (dense orbit) and attracting all nearby future orbits or at least a Lebesgue positive measure set in the ambient manifold. If A is an attractor for f with basin $B(A)$, we say that it is sensitive to initial conditions, or chaotic if there is $\varepsilon > 0$ such that with total probability on $B(A) \times B(A)$, for each pair of points (x, y) there is an integer $n > 0$ so that $f^n(x)$ and $f^n(y)$ are more than ε apart, where the distances are considered with respect to some Riemannian metric. The definition for flows is entirely similar. Chaotic attractors became also known as strange. The first one, beyond the hyperbolic attractors which are not just sinks, is due to Lorenz **[L]**. Proposed numerically by Lorenz in 1963, it's a rather striking fact that only in the middle seventies most of us became acquainted with Lorenz-like attractors through the examples of Guckenheimer and Williams, which we now call geometric ones. It is still an open and interesting question if the original Lorenz's equations

$$\dot{x} = -10x + 10y$$
$$\dot{y} = 28x - y - xz$$
$$\dot{z} = -(8/3)z + xy$$

in fact correspond to a flow displaying a strange attractor.

Subsequently, again based on numerical experiments, Hénon **[He]** asked about the possible existence of a strange attractor, but now in two dimensions, for certain quadratic diffeomorphisms of the plane

$$f_{a,b}(x, y) = (1 - ax^2 + y, bx)$$

for $a \approx 1.4$ and $b \approx 0.3$. Finally, by the end of the decade, Feigenbaum **[F]** and independently Coullet-Tresser **[CT]** suggested another kind of attractor, now for quadratic maps of the interval and related to a limiting map of a sequence of transformations exhibiting period-doubling bifurcations of periodic orbits. Almost immediately after that, Jacobson **[J]** exhibited strange attractors in the same setting. All this, together with the unsuccessful atempts of the sixties, led to a common belief that perhaps no such a global scenario for dynamics was possible.

However, a series of important results on strange attractors for maps, concerning their persistence, i.e. their existence for a positive Lebesgue measure set in parameter space, and the fact that they carry physical or SRB (Sinai-Ruelle-Bowen) invariant measures, provided by Jacobson **[J]** for the interval, Benedicks-Carleson **[BC]**, Mora-Viana **[MV]**, Benedicks-Young **[BY1]**, **[BY2]** and Diaz-Rocha-Viana **[DRV]** for Hénon-like maps (small C^r perturbations of Hénon maps, $r \geq 1$), were about to take place in the next fifteeen years or so. Perhaps even more striking is the recent proof that they are stochastically stable, a recent remarkable result of Benedicks-Viana **[BV1]**, **[BV2]**. In proving this fact, Benedicks and Viana first showed for Hénon-like attractors, that there are "no holes" in the basin of attraction with respect to the SRB measure, a question I heard from Ruelle and Sinai more than a decade ago: a.e. in the attractor with respect to the SRB measure, there are stable manifolds and their union covers Lebesgue a.a. points in the basin of attraction (and, thus, the union of the stable manifolds is dense in the basin of attraction) **[BV1]**. The concepts of SRB measure and stochastic stability will be presented below. Almost simultaneously, after previous pioneering work of Arnold and Herman (see **[Ar]**, **[H]**), the theory of one-dimensional dynamics experienced a great advance, due to Yoccoz, Sullivan, McMullen, Lyubich, Douady, Hubbard, Swiatek and an impressive number of other mathematicians (see **[dMS]** and **[dM]**).

Such developments, as well as my own work with Takens and Yoccoz, **[PT1]**, **[PT2]**, **[PY1]**, and many inspiring conversations with colleagues, former and present students, among them de Melo, Pujals, Takens, Yoccoz and above all Viana, made me progressively acquire a new global view of dynamics, emphasizing a much more