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HÖLDER IMPLIES COLLET-ECKMANN

by

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Abstract. — We prove that for every polynomial f if its basin of attraction to ∞ is Hölder and Julia set contains only one critical point c then f is Collet-Eckmann, namely there exists $\lambda > 1$, $C > 0$ such that, for every $n \geq 0$, $|(f^n)'(f(c))| \geq C\lambda^n$. We introduce also *topological Collet-Eckmann rational maps* and *repellers*.

0. Introduction

J. Graczyk and S. Smirnov proved in **[GS]** that if a rational map is Collet-Eckmann (abbr. CE), then every component of the complement of Julia set J is Hölder. Another proof was provided later in **[PR1]**. The question whether a converse fact holds remained unanswered. Moreover it has been proved in **[PR2]** (using an example from **[CJY]**) that if there are at least two critical points in J, then the converse may occur false, even for polynomials. Namely if the forward trajectory of a critical point c at some times approaches very closely another critical point, but all critical points in J are nonreccurrent, then A_∞ the basin of infinity is John even, but $|(f^n)'(f(c))|$ does not grow exponentially fast.

Here (in Sec.3) we prove that A_∞ Hölder implies CE for polynomials if there is only one critical point in J. In fact we prove this in a more general setting of rational functions. We prove this by using Graczyk and Smirnov's "reversed telescope" idea.

In Section 4 we introduce for rational maps the property *topological Collet-Eckmann* (abbr. TCE). This property means roughly a possibility of going from many small scales around each point to large scale round discs with uniformly bounded criticality

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under the action of the iterates of f. This property is topological (*i.e.* it is preserved under topological conjugacies) and we prove in Section 4 that it implies CE, provided there is only one critical point in J (or more than one, but none in the ω-limit set of the others). Since by **[PR1]** CE implies TCE we obtain a new elementary proof that CE is a topological property. The first proof was provided in **[PR2]**: For f being CE, and g topologically conjugate to it, it was proved that the conjugacy can be improved to a quasiconformal one on a neighbourhood of $J(f)$. This implied CE for g, by a method not much different from presented here (but simpler technically).

In the unimodal maps of the interval case the fact CE is a topological property was proved in **[NP]** via the same TCE property called there *finite criticality.* The intermediate property used there was *uniform hyperbolicity on periodic orbits* (abbr. UHPer). Here this idea also appears implicitly, though we cannot prove UHPer implies CE (the fact proved for unimodal maps of interval with negative Schwarzian derivative by T. Nowicki and D. Sands in **[NS]**.)

Finally, in Section 5, we introduce and study *holomorphic TCE invariant sets* in particular *repellers* and prove that if a repeller is the boundary of an open connected domain in $\overline{\mathbb{C}}$, then it is TCE iff the domain is Hölder. In consequence, for each domain with repelling boundary, to be Hölder is a topological property. We prove also the analogous rigidity result for Hölder immediate basins of attraction to attracting fixed points.

1. Preliminaries on Hölder basins

Definition 1.1. — Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational map of the Riemann sphere. We call an f-critical point c (*i.e.* such that $f'(c) = 0$) *exposed* if its forward f-trajectory does not meet other critical points.

The map f is called Collet-Eckmann if its every exposed critical point c that belongs to the Julia set $J = J(f)$, or its forward orbit converges to J, satisfies the following Collet-Eckmann condition:

There exists $\lambda > 1$ such that for every $n \geq 0$

$$(CE) \qquad |(f^n)'(c_1)| \geq \text{Const}\, \lambda^n.$$

Notation. — By Const we denote various positive constants which can change from one formula to another. We use the notation $x_n = f^n(x)$.

The definition of holomorphic Collet-Eckmann map was introduced in **[P2]** with (CE) assumed only for critical points in J. This allowed parabolic periodic points. Here we modify the definition, in accordance with **[GS**, Def 1.2].

One calls a simply-connected open hyperbolic domain A *Hölder* if there exists $\alpha > 0$ such that any Riemann mapping from the unit disc D onto A is Hölder continuous. This can be generalized to non-simply connected domains, see **[Po]** or **[GS**, Def 5.1].

We shall not rewrite here this definition in absence of dynamics because we do not need this. However if A is an immediate basin of attraction to a sink, for a rational mapping f, $f(A) = A$, Graczyk and Smirnov provided an equivalent definition [**GS**, Def.1.4, Sec.5 Prop.3] which will be of use for us. Denote $\mathrm{Crit}^+ := \bigcup_{j=1}^{\infty} f^j(\mathrm{Crit})$, where Crit= Crit($f$) means the set of all critical points for f.

Definition 1.2. — We call A Hölder if there exists $\lambda_{\mathrm{Ho}} > 1$ such that for every $z \in A \setminus \mathrm{cl}\,\mathrm{Crit}^+$ there exists $C_1 > 0$ such that for every $y \in f^{-n}(\{z\}) \cap A$

$$|(f^n)'(y)| \geq C_1 \lambda_{\mathrm{Ho}}^n. \tag{1.1}$$

We extend this definition to periodic A, $f^k(A) = A$, by replacing f by f^k above. This replacement allows in proofs to assume $f(A) = A$.

We need also the following

***Notation* (*cf.* [PUZ])**. — Suppose $f(A) = A$. Let $z^1, \ldots, z^d$ be all the pre-images of z in A. Consider smooth curves $\gamma^j : [0,1] \to A \setminus \mathrm{cl}\,\mathrm{Crit}^+$, $j = 1, \ldots, d$, joining z to z^j respectively (*i.e.* $\gamma^j(0) = z$, $\gamma^j(1) = z^j$).

Let $\Sigma^d := \{1, \ldots, d\}^{\mathbb{Z}^+}$ denote the one-sided shift space and σ the shift to the left, *i.e.* $\sigma((\alpha_n)) = (\alpha_{n+1})$. For every sequence $\alpha = (\alpha_n)_{n=0}^{\infty} \in \Sigma^d$ we define $\gamma_0(\alpha) := \gamma^{\alpha_0}$. Suppose that for some $n \geq 0$, for every $0 \leq m \leq n$, and all $\alpha \in \Sigma^d$, the curves $\gamma_m(\alpha)$ are already defined. Write $z_n(\alpha) := \gamma_n(\alpha)(1)$.

For each $\alpha \in \Sigma^d$ define the curve $\gamma_{n+1}(\alpha)$ as the lift (image) by $f^{-(n+1)}$ of $\gamma^{\alpha_{n+1}}$ starting at $z_n(\alpha)$.

The graph $\mathcal{T} = \mathcal{T}(z, \gamma^1, \ldots, \gamma^d)$ with the vertices z and $z_n(\alpha)$ and edges $\gamma_n(\alpha)$ is called a *geometric coding tree* with the root at z. For every $\alpha \in \Sigma^d$ the subgraph composed of $z, z_n(\alpha)$ and $\gamma_n(\alpha)$ for all $n \geq 0$ is called a *geometric branch* and denoted by $b(\alpha)$. Denote by $b_n(\alpha)$ for $n \geq 0$ the subgraph composed of $z_j(\alpha)$ and $\gamma_{j+1}(\alpha)$ for all $j \geq n$.

The branch $b(\alpha)$ is called *convergent* to $x \in \partial A$ if $z_n(\alpha) \to x$.

For an arbitrary basin of attraction A we define the *coding map* $z_\infty : \mathcal{D}(z_\infty) \to \mathrm{cl}\,U$ by $z_\infty(\alpha) := \lim_{n\to\infty} z_n(\alpha)$ on the domain $\mathcal{D} = \mathcal{D}(z_\infty)$ of all such α's for which $b(\alpha)$ is convergent. By Lemma 1.3 below, for A Hölder, $\mathcal{D} = \Sigma^d$ and z_∞ is Hölder.

Finally let $U^1, \ldots, U^d$ be open topological discs with closures in $A \setminus \mathrm{cl}\,\mathrm{Crit}^+$, containing $\gamma^1, \ldots, \gamma^d$ respectively. For each α and $n \geq 0$ denote by $U_n(\alpha)$ the component of $f^{-n}(U^{\alpha_n})$ containing $\gamma_n(\alpha)$.

In the subsequent Lemmas A is a Hölder immediate basin of attraction to a periodic sink for a rational function f.

Lemma 1.3. — *There exists $C_2 > 0$ such that for every $\alpha \in \Sigma^d$ and every positive integer m*

$$\mathrm{diam}\, U_m(\alpha) \leq C_2 \lambda_{\mathrm{Ho}}^{-m} \tag{1.2}$$

and

$$(1.3) \qquad U_m(\alpha) \subset B(z_\infty(\alpha), C_2\lambda_{\mathrm{Ho}}^{-m}/(1-\lambda_{\mathrm{Ho}}^{-1})).$$

Proof. — This follows from (1.1) and uniformly bounded distortion for all the branches of f^{-n}, $n \geq m$ on U^j involved. □

Lemma 1.4. — *For every $x \in \partial A$ there exists $\alpha \in \Sigma^d$ such that $b(\alpha)$ is convergent to x.*

Proof. — Notice that $x = \lim z_{n_k}(\widehat{\alpha}^k)$ for a sequence $\widehat{\alpha}^k \in \Sigma^d$ and a sequence of integers n_k, see [**PZ**, the proof of (9)]. Now any α a limit of a convergent subsequence of $\widehat{\alpha}^k$ satisfies the assertion of the Lemma. The convergence of $b(\alpha)$ is even exponential. This follows from Lemma 1.3 □

Lemma 1.5. — *Let A be a Hölder immediate basin of attraction to a sink for a rational map f. Then for every $\lambda : 1 < \lambda < \lambda_{\mathrm{Ho}}$ there exist $\delta > 0$ and $n_0 > 0$ such that for every $n \geq n_0$ and every $x \in \partial A$, if for every $j = 0, \dots, n-1$*

$$(1.4) \qquad \operatorname{dist}(x_j, \mathrm{Crit}) > \exp -\delta n,$$

then $|(f^n)'(x)| > \lambda^n$.

Proof. — Consider $\alpha \in \Sigma^d$ such that $b(\alpha)$ converges to x_0. Then for

$$s = [C_3 + n\delta/(\log \lambda_{\mathrm{Ho}})] + 1$$

(the square brackets stand for the integer part), where

$$C_3 = \frac{\left(\log C_2/\varepsilon(1-\lambda_{\mathrm{Ho}})\right)}{\log \lambda_{\mathrm{Ho}}},$$

for an arbitrary $\varepsilon : 0 < \varepsilon < 1$, one obtains by (1.3)

$$z_s(\sigma^n(\alpha)) \subset B(x_n, C_2\lambda_{\mathrm{Ho}}^{-s}/(1-\lambda_{\mathrm{Ho}}^{-1}) = B(x_n, \varepsilon \exp -\delta n).$$

Moreover for every $0 \leq j \leq n$

$$(1.5) \qquad z_{s+j}(\sigma^{n-j}(\alpha)) \in B(x_{n-j}, \varepsilon \exp -\delta n).$$

For $y := z_{s+n}(\alpha)$ we have

$$|(f^n)'(y)| = |(f^{n+s})'(y)| \cdot |(f^s)'(f^n(y))|^{-1} \geq C_1\lambda_{\mathrm{Ho}}^{n+s}L^{-s}$$

for $L := \sup |f'|$.

Using the definition of s we see that for δ small enough and n large, the latter expression is larger than $\tilde{\lambda}^n$ for an arbitrary $\tilde{\lambda} : 1 < \tilde{\lambda} < \lambda_{\mathrm{Ho}}$, so

$$(1.6) \qquad |(f^n)'(y)| > \tilde{\lambda}^n.$$

For ε small enough, in view of (1.4) and (1.5), we can replace y by x in (1.5), changing $\tilde{\lambda}$ by a factor arbitrarily close to 1. □