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*Astérisque*, tome 261 (2000), p. 57-88
<http://www.numdam.org/item?id=AST_2000__261__57_0>

*Astérisque*
**261**, 2000, p. 57–88

# COMPLEX FOLIATIONS WITH ALGEBRAIC LIMIT SETS

*by*

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*Dedicated to Adrien Douady on the occasion of his* $60^{\underline{th}}$ *birthday.*

***Abstract.*** — We regard the problem of classification for complex projective foliations with algebraic limit sets and prove the following:

*Let $\mathcal{F}$ be a holomorphic foliation by curves in the complex projective plane $\mathbb{C}P(2)$ having as limit set some singularities and an algebraic curve $\Lambda \subset \mathbb{C}P(2)$. If the singularities $\operatorname{sing}\mathcal{F} \cap \Lambda$ are generic then either $\mathcal{F}$ is given by a closed rational 1-form or it is a rational pull-back of a Riccati foliation $\mathcal{R} : p(x)dy - (a(x)y^2 + b(x)y)dx = 0$, where $\Lambda$ corresponds to $\overline{(y=0)} \cup \overline{(p(x)=0)}$, on $\overline{\mathbb{C}} \times \overline{\mathbb{C}}$.*

The proof is based on the solvability of the generalized holonomy groups associated to a reduction process of the singularities $\operatorname{sing}\mathcal{F} \cap \Lambda$ and the construction of an affine transverse structure for $\mathcal{F}$ outside an algebraic curve containing $\Lambda$.

## 1. Introduction

Let $\mathcal{F}$ be a holomorphic codimension one foliation on the complex projective 2-space $\mathbb{C}P(2)$. Given any leaf $L$ of $\mathcal{F}$ the *limit set* of $L$ is defined as $\lim(L) = \bigcap_\nu \overline{L \backslash K_\nu}$ where $K_\nu \subset K_{\nu+1}$ is an exhaustion of $L$ by compact subsets $K_\nu \subset L$. The *limit set* of the foliation $\mathcal{F}$ is defined as $\lim \mathcal{F} = \overline{\bigcup_L \lim(L)}$. We are interested in classifying those foliations whose limit set is a union of singularities of $\mathcal{F}$ and an algebraic curve $\Lambda \subset \mathbb{C}P(2)$. There are two reasons for this, first because these foliations exhibit the simplest dynamic behavior we can imagine and also because they must support an important class of first integrals. The parallel with the actions of Kleinian groups on the Riemann sphere comes naturally to mind. These foliations will correspond to actions with a finite set of limit points (one or two) while the first integrals of these foliations will correspond to the automorphic functions of such Kleinian group

**1991** ***Mathematics Subject Classification.*** — 32L30, 58F18.
***Key words and phrases.*** — Holomorphic foliation, limit set, holonomy.

B. Scárdua was supported by the CNPq-Brasil and the Ministère Affaires Étrangères (France).

actions. Here we will show that this similarity is not only apparent. Indeed the Kleinian groups will appear naturally as the holonomy groups of the Riccati foliation that, it will be shown here, is the ultimate model for these foliations.

The problem of classifying such foliations $\mathcal{F}$ was considered in **[1]** and **[17]**. In both cases it is proved that, under generic assumptions, there are a rational map $F\colon \mathbb{C}P(2) \to \mathbb{C}P(2)$ and a linear foliation $\mathcal{L} : \lambda_1 x dy - \lambda_2 y dx = 0$ on $\mathbb{C}P(2)$ such that $\mathcal{F} = F^*(\mathcal{L})$. In particular, it follows that no saddle-nodes appear in the resolution of $\operatorname{sing}\mathcal{F} \cap \Lambda$, and in fact all the singularities as well as all the holonomy groups appearing in this resolution are abelian and linearizable. Using **[9]** we can construct examples where $\mathcal{F}$ is a *Riccati foliation* with algebraic limit set on $\mathbb{C}P(2)$, containing the invariant line $\overline{(y=0)}$:

$$\mathcal{F} : p(x)dy - (y^2 a(x) + y b(x))dx = 0$$

where $a(x), b(x), p(x)$ are polynomials, and $(x,y) \in \mathbb{C}^2 \subset \mathbb{C}P(2)$ is an affine chart (see Example 1.3 below) and $\Lambda \cap \mathbb{C}^2 = (p(x) = 0) \cup (y = 0)$.

In the Riccati case, the holonomy groups are solvable and we have an additional compatibility condition as in **[2]**. However we may have saddle-nodes in the resolution of $\operatorname{sing}\mathcal{F} \cap \Lambda$. The aim of this paper is to solve the problem above in the case the foliation may have certain saddle-node singularities in its resolution along $\Lambda$.

Let therefore $\mathcal{F}$ be a foliation on $\mathbb{C}P(2)$ and let $\Lambda \subset \mathbb{C}P(2)$ be an algebraic invariant curve (perhaps reducible). We will say that $\operatorname{sing}\mathcal{F} \cap \Lambda$ *has* the pseudoconvexity property (*psdc*) if the *invariant (by $\mathcal{F}$) part* $\Gamma$ of the resolution divisor $D$ of $\operatorname{sing}\mathcal{F} \cap \Lambda$ is connected and its complement is a *Stein manifold* (alternatively, $\Gamma$ is a very ample divisor on the ambient (algebraic) manifold of the resolution of $\operatorname{sing}\mathcal{F} \cap \Lambda$ denoted by $\widetilde{\mathbb{C}P(2)}$), so that we can apply Levi's extension theorem **[21]** which allows us to extend analytically to all $\mathbb{C}P(2)$, any analytic object defined on a neighborhood of $\Gamma$. This property is verified if $\mathcal{F}$ has no dicritical singularities over $\Lambda$ **[1]**. There is another remarkable case where property (*psdc*) is verified, as we can find in **[17]**. A singularity $q_o \in \operatorname{sing}\widetilde{\mathcal{F}} \cap D$ is a *corner* if $q_o = D_i \cap D_j$, where $D_i \neq D_j$ are invariant components of $D$.

Also, we say that a saddle-node singularity $q_o \in D$ is in *good position* relatively to $D$, if its *strong separatrix* is contained in some component of $\Gamma$. A saddle-node $x^{k+1}dy - y(1+\lambda x^k)dx + \text{h.o.t.} = 0$ is *analytically normalizable* if we may choose local coordinates $(x,y)$ as above for which we have $\text{h.o.t.} = 0$. In this case it will be called *normally hyperbolic* if we have $\lambda \notin \mathbb{Q}$. In this case we call $(x=0)$ the *strong separatrix* and $(y=0)$ the *central manifold* of the saddle-node. We recall that according to **[12]** a saddle-node singularity is analytically classified by the local holonomy of this strong separatrix. In particular, the saddle-node is analytically normalizable if, and only if, its strong separatrix holonomy is an analytically normalizable flat diffeomorphism.

Finally, we introduce the following technical condition (see Example 1.4):

$(C_1)$ *The saddle-nodes in the resolution of* $\operatorname{sing}\mathcal{F}\cap\Lambda$ *are analytically normalizable, and the ones in the corners are normally hyperbolic.*

Our main result is the following:

***Theorem 1.1***. — *Let* $\mathcal{F}$ *be a codimension one holomorphic foliation on* $\mathbb{C}P(2)$ *having as limit set some singularities and an algebraic curve* $\Lambda\subset\mathbb{C}P(2)$. *Assume that* $\operatorname{sing}\mathcal{F}\cap\Lambda$ *satisfies property (psdc) and condition* $C_1$. *Then, either* $\mathcal{F}$ *is given by a closed rational* 1*-form or it is a rational pull-back of a Riccati foliation* $\mathcal{R}: p(x)dy-(a(x)y^2+b(x)y)dx=0$, *where* $\Lambda$ *corresponds to* $\overline{(y=0)}\cup\overline{(p(x)=0)}$, *on* $\overline{\mathbb{C}}\times\overline{\mathbb{C}}$.

The proof of this theorem relies on the study of the *singular* and *virtual* holonomy groups **[2]**, **[5]**, **[19]** and **[1]** respectively, of the irreducible components of the divisor given by the resolution of $\operatorname{sing}\mathcal{F}\cap\Lambda$. *The limit set of the leaves* $\widetilde{L}$ *of* $\widetilde{\mathcal{F}}$ *induces discrete pseudo-orbits in each of these groups, so that they are solvable* **[14]**. The solvability of these groups, allows (under our restrictions on $\operatorname{sing}\mathcal{F}\cap\Lambda$) the construction of a "transversely formal" meromorphic 1-form $\widetilde{\eta}$, defined over the invariant part $\Gamma$ of the resolution divisor of $\operatorname{sing}\mathcal{F}\cap\Lambda$. This 1-form is closed and satisfies the relation $d\widetilde{\omega}=\widetilde{\eta}\wedge\widetilde{\omega}$, where $\widetilde{\omega}$ is a meromorphic 1-form with isolated singularities which defines the foliation $\widetilde{\mathcal{F}}$, obtained from the resolution of $\operatorname{sing}\mathcal{F}\cap\Lambda$. Moreover, $\widetilde{\eta}$ has (simple) poles over $\Gamma$ which coincides with the limit set of $\widetilde{\mathcal{F}}$. Using a result of Hironaka-Matsumara (see **[5]**, **[8]**), we conclude that (since $\widetilde{\mathbb{C}P(2)}\backslash\Gamma$ is a Stein manifold) the 1-form $\widetilde{\eta}$ is in fact rational on $\widetilde{\mathbb{C}P(2)}$. This corresponds to the existence of a Liouvillian first integral for $\mathcal{F}$ on $\mathbb{C}P(2)$, and also to the existence of an affine transverse structure for $\widetilde{\mathcal{F}}$ in $\widetilde{\mathbb{C}P(2)}\backslash C$, where $C\subset\widetilde{\mathbb{C}P(2)}$ is an algebraic invariant curve containing $\widetilde{\Lambda}$, where $\widetilde{\Lambda}$ is the strict transform of $\Lambda$, **[18]**. This affine transverse structure can be extended as a projective transverse structure to $(\widetilde{\mathbb{C}P(2)}\backslash C)\cup\widetilde{\Lambda}$. In particular, all the singular holonomy groups associated to the components of $\Gamma$ are solvable analytically normalizable. This implies by (a careful reading of the last part of) **[2]** that either $\mathcal{F}$ is given by a closed rational 1-form or by a rational pull-back of a Riccati foliation.

***Example 1.2***. — Let $\mathcal{F}$ be a rational pull-back of a hyperbolic linear foliation $\mathcal{L}$ : $xdy-\lambda ydx=0, \lambda\in\mathbb{C}\backslash\mathbb{R}$, on $\mathbb{C}P(2)$. Clearly $\mathcal{F}$ has an algebraic limit set consisting of some singularities and an algebraic curve $\Lambda$ as in Theorem 1.1.

***Example 1.3***. — Let us take any finitely generated group of Moebius transformations $G\subset \mathrm{SL}(2,\mathbb{C})$. Assume that the limit set of $G$ is a single point, which can be assumed to be the origin $0\in\overline{\mathbb{C}}$. The limit point 0 is a fixed point of $G$. According to **[9]** we can find a Riccati foliation $\mathcal{F}: p(x)dy-(a(x)y^2+b(x)y+c(x))dx=0$ on $\overline{\mathbb{C}}\times\overline{\mathbb{C}}$, whose holonomy group of the line $\overline{(y=0)}$ is conjugated to the group $G$. Moreover we can assume that the singularities of $\mathcal{F}$ over this horizontal line are reduced and non degenerate. The line $\overline{(y=0)}$ is invariant by $\mathcal{F}$ so that $c(x)=0$, and also it is contained in the limit set of $\mathcal{F}$ and satisfies condition $C_1$ in the statement above. This example can also be seen in $\mathbb{C}P(2)$ using a birational transformation. This will create a dicritical singularity. This example will satisfy the (*psdc*) property for a proper choice of $\Lambda$.

***Example 1.4***. — This is a counterexample to a more general statement. Let $\mathcal{F}$ be given by $\omega = dy - (a(x)y + b(x))dx = 0$ over $\mathbb{C}^2 \subset \mathbb{C}P(2)$. If we consider the vector field $X(x,y) = (1, a(x)y + b(x))$, then $X$ is complete and tangent to $\mathcal{F}$ over $\mathbb{C}^2$. Moreover the orbits of $X$ are diffeomorphic to $\mathbb{C}$. It is not difficult to see, using the flow of $X$, that the leaves of $\mathcal{F}$ accumulate the line at infinity $L_\infty = \mathbb{C}P(2)\backslash\mathbb{C}^2$, so that $\lim \mathcal{F} = L_\infty$. However, generically, the resolution of $\operatorname{sing}\mathcal{F} \cap L_\infty$ exhibits some non analytically normalizable saddle-node. Indeed, this resolution is quite simple and shows that there are saddle-nodes with non convergent central manifolds [**5**]. On the other hand, in general, $\mathcal{F}$ is not a rational pull-back of a Riccati foliation of the form stated in Theorem 1.1.

*Acknowledgements*. — Part of this work was conceived during a post-doctorade stage of the second author, at the Université de Rennes I. He wants to thank the IRMAR and specially D. Cerveau for the kind hospitality and for valuable conversations.

## 2. Formal normal forms and resolution of singularities

Let $\mathcal{F}$ and $\Lambda \subset \lim \mathcal{F}$ be as in Theorem 1.1. Let $\pi\colon (\widetilde{\mathbb{C}P(2)}, \widetilde{\mathcal{F}}, D) \to (\mathbb{C}P(2), \mathcal{F}, \Lambda)$ be the resolution morphism of Seidenberg, for $\operatorname{sing}\mathcal{F} \cap \Lambda$ [**20**]. Thus $\widetilde{\mathbb{C}P(2)}$ is a compact complex surface which is obtained from $\mathbb{C}P(2)$ by a finite sequence of blowing-up's, denoted $\pi$. The proper morphism $\pi$ induces therefore a foliation by curves $\widetilde{\mathcal{F}} = \pi^*\mathcal{F}$ on $\widetilde{\mathbb{C}P(2)}$. The divisor $D = \pi^{-1}(\Lambda)$ of the resolution is a finite union $D = \bigcup_{j=0}^m D_j$, of projective lines $D_j \cong \mathbb{C}P(1)$, $j \neq 0$, and of the strict transform of $\Lambda$, $D_0 = \overline{\pi^{-1}(\Lambda\backslash \operatorname{sing}\mathcal{F})}$. The foliation $\widetilde{\mathcal{F}}$ has singularities of the following two types (called *irreducible* singularities):

(i) $xdy - \lambda ydx + \text{h.o.t.} = 0$ (non degenerate)
(ii) $y^{p+1}dx - [x(1 + \lambda y^p) + \text{h.o.t.}]dy = 0$ (saddle-node).

We consider the foliation $\widetilde{\mathcal{F}} = \pi^*\mathcal{F}$ and denote by $\Gamma$ the invariant (by $\widetilde{\mathcal{F}}$) part of $D$, which consists of the invariant projective lines and of the strict transform of $\Lambda$. Let $\omega$ be a rational 1-form which defines $\mathcal{F}$ on $\mathbb{C}P(2)$ and denote by $\widetilde{\omega}$ the strict transform of $\pi^*\omega$. Therefore the 1-form $\widetilde{\omega}$ has isolated singularities and we can assume that its polar set intersects the divisor $D$ transversely and at regular points of $\widetilde{\mathcal{F}}$. Clearly we have $\lim(\widetilde{\mathcal{F}}) \subset \Gamma$.

***Lemma 2.1***. — *We have* $\lim(\widetilde{\mathcal{F}}) = \Gamma$. *In particular all the saddle-nodes in* $\operatorname{sing}\widetilde{\mathcal{F}} \cap \Gamma$ *are in good position with respect to* $\Gamma$.