Astérisque
273, 2001, p. 13–80

BERNSTEIN DEGREE AND ASSOCIATED CYCLES OF HARISH-CHANDRA MODULES – HERMITIAN SYMMETRIC CASE –

by

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Dedicated to Professor Ryoshi Hotta on his 60th anniversary

Abstract. — Let $\tilde{G}$ be the metaplectic double cover of $Sp(2n, \mathbb{R})$, $U(p, q)$ or $O^*(2p)$. we study the Bernstein degrees and the associated cycles of the irreducible unitary highest weight representations of $\tilde{G}$, by using the theta correspondence of dual pairs. The first part of this article is a summary of fundamental properties and known results of the Bernstein degrees and the associated cycles. Our first result is a comparison theorem between the K-module structures of the following two spaces; one is the theta lift of the trivial representation and the other is the ring of regular functions on its associated variety. Secondarily, we obtain the explicit values of the degrees of some small nilpotent $K_{\mathbb{C}}$-orbits by means of representation theory. The main result of this article is the determination of the associated cycles of singular unitary highest weight representations, which are the theta lifts of irreducible representations of certain compact groups. In the proofs of these results, the multiplicity free property of spherical subgroups and the stability of the branching coefficients play important roles.

***Résumé* (Le degré de Bernstein et le cycle associé des modules de Harish-Chandra — le cas hermitien symétrique)**

Soit $\tilde{G}$ le revêtement double métaplectique de $Sp(2n, \mathbb{R})$, $U(p, q)$ ou $O^*(2p)$. Nous étudions les degrés de Bernstein et les cycles associés des représentations irréductibles unitaires de $\tilde{G}$ de plus haut poids, en utilisant la correspondance thêta par paires duales. La première partie de cet article est un résumé des propriétés fondamentales et des résultats connus concernant les degrés de Bernstein et les cycles associés. Notre premier résultat est un théorème de comparaison entre les structures en tant que K-modules des deux espaces suivants : l'un est le relèvement thêta de la représentation évidente, l'autre est l'anneau des fonctions régulières sur la variété associée. Deuxièmement, nous obtenons de manière explicite les valeurs des degrés de quelques petites $K_{\mathbb{C}}$-orbites nilpotentes au moyen de la théorie des représentations. Le résultat principal de cet article est la détermination des cyles associés aux représentations singulières unitaires de plus haut poids, qui sont les relèvements thêta des représentations irréductibles de certains groupes compacts. Dans les démonstrations de ces résultats, la non-multiplicité des sous-groupes sphériques et la stabilité des coefficients de branchement jouent des rôles importants.

2000 ***Mathematics Subject Classification.*** — Primary 22E46, 32M15, 14L30, 58J70.
Key words and phrases. — Hermitian symmetric space, highest weight representation, nilpotent orbit, associated variety, associated cycle, multiplicity-free action, dual pair correspondence.

Introduction

Let G be a semisimple (or more generally, reductive) Lie group. For an irreducible admissible representation π of G, there exist several important invariants such as irreducible characters, primitive ideals, associated varieties, asymptotic supports, Bernstein degrees, Gelfand-Kirillov dimensions, etc. They are interrelated with each other, and intimately related to the geometry of coadjoint orbits.

For example, at least if G is compact and π is finite dimensional, the character of π is the Fourier transform of an orbital integral on a semisimple coadjoint orbit (**[29]**). This is also the case for a general semisimple G and fairly large family of the representations (see **[41]**). This intimate relation between coadjoint orbits and irreducible representations invokes the philosophy of so-called *orbit method*, which is exploited by pioneer works of Kirillov and Kostant, and is now being developed by many contributors. However, for a general semisimple Lie group G, it seems that the orbit method still requires much to do. In particular, we should understand some small representations corresponding to nilpotent coadjoint orbits, which are called *unipotent*.

On the other hand, by definition, most of invariants are directly related to *nilpotent* coadjoint orbits. In a sense, the corresponding nilpotent orbits represent the leading term of irreducible characters (**[1]**, **[44]**). The invariants of large representations correspond to the largest nilpotent coadjoint orbit, namely, the principal nilpotent orbit. For large representations, the orbit method seems to behave considerably well. Therefore we are now interested in 'small' representations whose invariants are related to smaller nilpotent coadjoint orbits.

One extreme case is the case of finite dimensional representations. In this case, however, the corresponding orbit is zero, and there is not a so much interesting phenomenon. The next to the extreme case is the case of minimal representations, which corresponds to the minimal nilpotent orbit. The minimal nilpotent orbit is unique in the sense that it is the only orbit among non-zero nilpotent ones with the smallest possible dimension. These representations have a simple structure. For example, their K-type structure is in a ladder form and is multiplicity free (**[50]**). Against its simple structure, though, systematic and thorough study of the minimal representations is still progressing now through the works of Kostant-Brylinski and many other mathematicians. If we turn our attention to the small representations other than minimal ones, it seems that there is relatively less knowledge on them up to now. In this paper, we study small representations which are unitary lowest (or highest) weight representations of G. Such representations exist if and only if G/K enjoys a structure of Hermitian symmetric space, where K denotes a maximal compact subgroup of G.

To be more specific, let us introduce notations. We assume that the symmetric space G/K is irreducible and Hermitian. Moreover, we assume that G is classical other than $SO(n,2)$, i.e., $G = Sp(2n,\mathbb{R}), U(p,q)$ or $O^*(2p)$. Let $\mathfrak{g}_0$ be the Lie algebra of G and $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ the Cartan decomposition with respect to K. We denote the complexified decomposition by $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. Since G/K is an irreducible Hermitian symmetric space, the induced adjoint representation of K on $\mathfrak{p}$ breaks up into precisely two irreducible components $\mathfrak{p} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$. Note that, as a representation of K, $\mathfrak{p}^-$ is contragredient to $\mathfrak{p}^+$ via the Killing form. We extend this representation of K to the representation of the complexification $K_{\mathbb{C}}$ of K holomorphically.

Let L be an irreducible unitary lowest weight module of G. Then it is well-known that the associated variety of L, denoted by $\mathcal{AV}(L)$, is the closure of a single nilpotent $K_{\mathbb{C}}$-orbit contained in $\mathfrak{p}^-$ (we choose an appropriate positive system which is compatible with $\mathfrak{p}^+$).

Put $r = \mathbb{R}\text{-rank}\, G$, the real rank of G. Then there exist exactly $(r+1)$ nilpotent $K_{\mathbb{C}}$-orbits $\{\mathcal{O}_0, \mathcal{O}_1, \ldots, \mathcal{O}_r\}$ in $\mathfrak{p}^-$. We choose an indexing of the orbits so that $\dim \mathcal{O}_{i-1} < \dim \mathcal{O}_i$ holds for $1 \leqslant i \leqslant r$; in particular, $\mathcal{O}_0 = \{0\}$ is the trivial one, and $\mathcal{O}_r$ is the open dense orbit. Most of lowest weight representations L correspond to the largest orbit $\mathcal{O}_r$. For example, the associated variety of a holomorphic discrete series (or its limit) is $\overline{\mathcal{O}_r} = \mathfrak{p}^-$. The invariants of the holomorphic discrete series representations are completely understood (see **[14]**, **[43]**, **[7]**; also see § 2.4 below). However, for each orbit $\mathcal{O}_m$ $(0 < m < r)$, there exists a relatively small family of lowest weight representations whose associated variety is indeed the closure of the orbit $\mathcal{O}_m$. Thanks to the theory of reductive dual pairs via the Weil representation of metaplectic groups, we have a complete knowledge of such a family of lowest weight representations (at least for classical groups listed above).

Although we can define a specific 'small' representations even for the largest orbit $\mathcal{O}_r$, we restrict ourselves to the case $\mathcal{O}_m$ $(m < r)$ in this introduction. Then there exists a compact group G_2 corresponding to each m (cf. § 3, Table 2) such that (G_1, G_2) forms a dual pair in a large symplectic group $Sp(2N, \mathbb{R})$. Let $Mp(2N, \mathbb{R})$ be the metaplectic double cover of $Sp(2N, \mathbb{R})$. We denote by $\widetilde{H} \subset Mp(2N, \mathbb{R})$ the inverse image of a subgroup $H \subset Sp(2N, \mathbb{R})$ of the covering map.

The family of unitary irreducible lowest weight representations of $\widetilde{G}$ whose associated variety is $\overline{\mathcal{O}_m}$ is parametrized by $\mathrm{Irr}(G_2)$, the set of the irreducible finite dimensional representations of G_2. We denote the lowest weight representation of $\widetilde{G}$ corresponding to $\sigma \in \mathrm{Irr}(G_2)$ by $L(\sigma)$ (see § 5 for precise description). Roughly, the correspondence $\sigma \mapsto L(\sigma)$ is the theta lift after twisted by a certain unitary character of $\widetilde{G_2}$.

Our first observation is the following.

Theorem A. — *Let $\mathbf{1}_{G_2}$ be the trivial representation of G_2 and $L(\mathbf{1}_{G_2})$ the unitary lowest weight representation of $\widetilde{G}$ corresponding to $\mathbf{1}_{G_2}$. The Bernstein degree of $L(\mathbf{1}_{G_2})$ coincides with the degree of the closure of the nilpotent orbit $\overline{\mathcal{O}_m}$ (defined in the sense of algebraic geometry) ;*

$$\operatorname{Deg} L(\mathbf{1}_{G_2}) = \deg \overline{\mathcal{O}_m}.$$

We also get an explicit and computable formula for $\operatorname{Deg} L(\mathbf{1}_{G_2})$.

Note that the varieties $\overline{\mathcal{O}_m}$ are determinantal varieties of various type and an explicit formula of their degree is known as *Giambelli-Thom-Porteous formula.* Our representation theoretic proof of the formula seems new, and gives an alternative proof.

To prove Theorem A, we construct a $K_{\mathbb{C}}$-equivariant map $\psi : V \to \overline{\mathcal{O}_m}$, where V is a certain $K_{\mathbb{C}} \times (G_2)_{\mathbb{C}}$-module. This map induces an algebra isomorphism

$$\psi^* : \mathbb{C}[\overline{\mathcal{O}_m}] \xrightarrow{\sim} \mathbb{C}[V^*]^{(G_2)_{\mathbb{C}}},$$

which means that $\overline{\mathcal{O}_m} = V//(G_2)_{\mathbb{C}}$. The map ψ is closely related to the dual pair (G, G_2), and we call it *unfolding* of $\overline{\mathcal{O}_m}$. By this, the proof of Theorem A reduces to a problem of classical invariant theory.

The 'smallest' unipotent representation attached to the orbit $\mathcal{O}_m$ should be realized on the section of a certain line bundle on $\mathcal{O}_m$ called half-form bundle (**[5]**, **[6]**, **[52]**). We investigate such half-form bundles, and get an evidence of strong relationship between the space of global sections of the half-form bundles and $L(\sigma)$, where σ is a special one-dimensional character of G_2.

Next, let us consider a general unitary lowest weight module $L(\sigma)$ ($\sigma \in \mathrm{Irr}(G_2)$). We describe its K-type decomposition and the Poincaré series in terms of certain branching coefficient of finite dimensional representations of general linear groups and G_2. Such descriptions are well-known among experts. However, references to them are scattered in many places, and sometimes their treatments are ad hoc. Since we need an explicit and unified picture for the K-types of $L(\sigma)$, we reproduce the decompositions in the sequel.

Now our main theorem says

Theorem B. — *Let $L(\sigma)$ be an irreducible unitary lowest weight module of $\widetilde{G}$ corresponding to $\sigma \in \mathrm{Irr}(G_2)$. Then its Bernstein degree is given by*

$$\mathrm{Deg}\, L(\sigma) = \dim \sigma \cdot \deg \overline{\mathcal{O}_m}.$$

There is a notion of *associated cycle* which is a refinement of the notion of associated variety. Roughly speaking, it expresses associated variety with multiplicity. For a precise definition, see §§ 1.1 and 1.3. Then the following is an immediate corollary to Theorem B.

Theorem C. — *The associated cycle of $L(\sigma)$ is given by $\mathcal{AC}\,(L(\sigma)) = \dim \sigma \cdot [\overline{\mathcal{O}_m}]$.*

The proof of Theorem B is based on the theory of multiplicity free action of algebraic groups, which is a subject of § 8. The key ingredients of the proof are multiplicity free property of spherical subgroups and Sato's summation formula of the stable branching coefficients.

Lastly, we would like to comment on several aspects of our results.

First, the Bernstein degree of an irreducible representation π is closely related to the dimension of its "Whittaker vectors". In fact, for large representations, Matumoto proved that the Bernstein degree and the dimension of algebraic Whittaker vectors coincide ([**36**]). For 'small' representations, we cannot hope the same story, because they do not have any Whittaker vector in a naive sense. However, for complex semisimple Lie groups, Matumoto observed that the finite-dimensionality and non-vanishing of the space of certain degenerate Whittaker vectors determines the wave front set of π ([**34**], [**35**]). Recently, Yamashita has found a strong relation between the multiplicity of associated cycles and the dimension of generalized Whittaker vectors in the case of unitary highest weight module ([**54**]).

Second, let us consider the (twisted) theta correspondence (or Howe correspondence, dual pair correspondence, ...) between $L(\sigma) \in \operatorname{Irr}(\widetilde{G})$ and $\sigma \in \operatorname{Irr}(G_2)$. Since G_2 is compact and σ is finite dimensional, its associated cycle is simply given by $\mathcal{AC}(\sigma) = \dim \sigma \cdot [\{0\}]$. Recall $\mathcal{AC}(L(\sigma)) = \dim \sigma \cdot [\overline{\mathcal{O}_m}]$ from Theorem C. These formulas strongly indicate the following; there should be a correspondence between nilpotent orbits of the dual pairs, and it induces certain relation between associated cycles of representations in theta correspondence. An optimistic reflection suggests that, if $L(\sigma)$ is a theta lift of σ, then their associated varieties are related as

$$\mathcal{AC}(L(\sigma)) = \sum_i m_i[\overline{\mathcal{O}_i}] \quad \longleftrightarrow \quad \mathcal{AC}(\sigma) = \sum_i m_i[\overline{\mathcal{O}'_i}],$$

with the same multiplicity, where $\mathcal{O}_i \leftrightarrow \mathcal{O}'_i$ indicates the orbit correspondence. However we do not have an intuitive evidence of such a kind of correspondence other than the cases treated here.

Third, Theorem A (or K-type decompositions) suggests that we should "quantize" the orbit $\mathcal{O}_m$ to get an irreducible unitary representation $L(\mathbf{1}_{G_2})$, which certainly should be a unipotent representation. For this, it will be helpful to try the similar method exploited by Kostant-Brylinski in the case of the minimal orbit. However, this will require much more than what we have presented in this note.

Now let us explain each section briefly.

In § 1, we define the associated cycles and other important invariants of representations in a general setting. After that, we collect their basic properties which will be needed later. In particular, in Lemma 1.1 and Theorem 1.4, we clarify the relationship between the associated cycles and the Bernstein degree (or the degree of the projectivised nilpotent cone); also, we recall the fact that the associated variety is the projection of the characteristic variety under the moment map (Lemma 1.6).

In § 2, we briefly summarize known facts and examples of associated cycles of various types of representations. To see what is going on in this paper, §§ 1.3 and 2.4 will be extremely useful.

In § 3, we review the properties of a reductive dual pair which we will need later. After an explicit description of the Fock realization of the Weil representation in