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THE DEGREES OF ORBITS OF THE MULTIPLICITY-FREE ACTIONS

by

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Abstract. — We give a formula for the degrees of orbits of the irreducible representations with multiplicity-free action. In particular, we obtain the Bernstein degree and the associated cycle of the irreducible unitary highest weight modules of the scalar type for arbitrary hermitian Lie algebras.

***Résumé* (Degrés des orbites nilpotentes des représentations irréductibles sans multiplicité)**
Nous donnons une formule pour les degrés des orbites nilpotentes des représentations irréductibles sans multiplicité. Nous obtenons les degrés de Bernstein et les cycles associés des représentations irréductibles unitaires de plus haut poids de type scalaire pour des algèbres de Lie hermitiennes.

1. Introduction

Let K be a connected reductive complex algebraic group, and V an irreducible representation of K. We assume that the action of K is multiplicity-free; that is, each irreducible representation of K occurs at most once in the polynomial ring $\mathbb{C}[V]$. We also assume that the image of K in $GL(V)$ contains all nonzero scalar matrices $\mathbb{C}^{\times}\,\mathrm{id}_V$. Such representations have been classified by Kac **[10]**. There are eight families and five exceptional representations.

In this paper, we determine the degree of each closed K-stable subset Y of V. We establish a method by which we can express some asymptotic behavior of the dimension of the filtered module in terms of a definite integral. This is a generalization of the technique presented in Ref. **[19]**. As a corollary, a formula for the degree of each K-stable closed subset can be obtained (Theorem 2.5). The multiplicity-free action contains an important family coming from the hermitian symmetric spaces. Such representations consist of four families and two exceptionals of the classification

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mentioned above. Using the detailed structure of the restricted root system, we can obtain a formula in these hermitian symmetric cases that is more concise than that obtained in the general case (Theorem 3.2). This formula unifies three kinds (i.e., homomorphism, symmetric endomorphism and skew-symmetric endomorphism) of Giambelli formulas, as well as the corresponding formula for the exceptional Lie algebras. The formula for the degree of the closure of the orbit immediately gives the Bernstein degree of the irreducible unitary highest weight module of the scalar type(Corollary 4.1). For three families of classical Lie algebras $\mathfrak{sp}(n, \mathbb{R})$, $\mathfrak{u}(p, q)$ and $\mathfrak{o}^*(2n)$, this result is obtained in Section 7 of Ref. **[19]** through case analysis. In the final section, we give two examples demonstrating the calculation of the Bernstein degree of the unitary highest weight modules of the non-scalar type. These are also derived from Theorem 2.3. In the Appendix, we list the explicit values for the degree of the closure of the orbits for all thirteen families of multiplicity-free actions, with some comment on the structure of the orbits.

A part of this paper is taken from the master thesis of the first author **[12]**.

2. Degree of the multiplicity-free action

2.1. Degree. — Let V be a finite-dimensional complex vector space, $\mathbb{C}[V]$ the ring of polynomials on V, and M a finitely-generated $\mathbb{C}[V]$-module. By a standard procedure, we can associate two additive, numerical invariants, the dimension and the multiplicity of M. This procedure is briefly summarized in Section 1 of Ref. **[19]** in this volume.

Let Y be a closed conic subvariety of V, and let $\mathbf{I}(Y)$ be the defining ideal of Y;

$$\mathbf{I}(Y) = \{p \in \mathbb{C}[V] \mid p(y) = 0 \text{ for all } y \in Y\}.$$

We define $\mathbb{C}[Y] = \mathbb{C}[V]/\mathbf{I}(Y)$. Defined in this manner, $\mathbb{C}[Y]$ is the coordinate ring of Y. Since $\mathbf{I}(Y)$ is a (reduced) graded ideal of $\mathbb{C}[V]$, $\mathbb{C}[Y]$ is naturally a graded $\mathbb{C}[V]$-module. The multiplicity of $\mathbb{C}[Y]$ is called the degree of Y, and is denoted by $\deg(Y)$. It is known that the degree of a complete intersection is elementary.

Lemma 2.1

(i) *If Y is a complete intersection, then the degree of Y is the product of the degrees of the defining equations of the irreducible components of Y.*

(ii) *If Y is a hypersurface, then the degree of Y is the degree (as a homogeneous polynomial) of the defining equation of Y.*

(iii) *If Y is a linear subspace of V, then the degree of Y is* 1.

The assertion (iii) is a special case of (ii), and (ii) is a special case of (i). The assertion (i) is found in standard textbooks, such as Ref. **[4]**. On the other hand, if the variety Y is not a complete intersection, such as a determinantal variety, its degree is non-trivial, as can be seen from the Giambelli formula.

2.2. Asymptotic behavior of some graded module. — Let K be a connected reductive complex algebraic group, and let V be a finite dimensional representation of K. Let $\mathbb{C}[V]^i$ be the set of homogeneous polynomials in $\mathbb{C}[V]$ of degree i. We assume that the image of K in $GL(V)$ contains all nonzero scalar matrices. Then, there exists an element $Z \in \mathrm{Lie}(K)$ such that $Z \cdot p = ip$ for all $p \in \mathbb{C}[V]^i$. This element is called the degree operator (or Euler operator). We denote the natural action of K on the graded algebra $\mathbb{C}[V]$ by Ad. We call M a $(\mathbb{C}[V], K)$-module if M is a $\mathbb{C}[V]$-module and is a completely reducible K-module with the compatibility condition $k \cdot (p \cdot (k^{-1} \cdot m)) = (\mathrm{Ad}\,(k)(p)) \cdot m$ for all $k \in K$, $p \in \mathbb{C}[V]$ and $m \in M$. We denote the decomposition into K-isotypic components by $M = \oplus_\mu M_\mu$. Assume that there exists some isotypic component M_λ generating M as a $\mathbb{C}[V]$-module. Such a component is unique if it exists. We define a graded component by $M^i = \mathbb{C}[V]^i M_\lambda$ for $i \in \mathbb{Z}_{\geqslant 0}$. Then $M = \oplus_i M^i$ is a graded $\mathbb{C}[V]$-module, and each graded component is given by

$$M^i = \{m \in M \mid Z \cdot m = (\lambda(Z) + i)m\}.$$

We assume, moreover, that M has a multiplicity-free decomposition

$$M = \bigoplus_{\varphi \in \Lambda(M)} F(\lambda + \varphi),$$

where $F(\mu)$ is a (finite-dimensional) irreducible K-module whose highest weight is μ, and that there exists linearly independent weights $\varphi_1, \dots, \varphi_m$ such that

$$\Lambda(M) = \{n_1\varphi_1 + \cdots + n_m\varphi_m \mid n_i \in \mathbb{Z}_{\geqslant 0}\}.$$

In this case, the graded component M^i is given by

$$M^i = \oplus F(\lambda + n_1\varphi_1 + \cdots + n_m\varphi_m),$$

where the summation is over $(n_1, \dots, n_m) \in \mathbb{Z}^m_{\geqslant 0}$ with $n_1\varphi_1(Z) + \cdots + n_m\varphi_m(Z) = i$. We will determine the asymptotic of the dimension of the graded component for large i.

Using the Weyl dimension formula, it can be shown that $\dim F(\lambda + n_1\varphi_1 + \cdots + n_m\varphi_m)$ is a polynomial in $(n_1, \dots, n_m)$. To be more explicit, let Δ_K^+ be the set of positive roots of the Lie algebra of K, and let ρ_K be the half sum of positive roots. We define

$$\Delta_M^+ = \Delta_K^+ \setminus \{\alpha \in \Delta_K^+ \mid \langle \alpha, \varphi_i \rangle = 0 \text{ for all } i = 1, \dots, m\}$$

and

$$\begin{aligned} f(x_1, \dots, x_m) = &\prod_{\alpha \in \Delta_K^+ \setminus \Delta_M^+} \frac{\langle \alpha, \lambda + \rho_K \rangle}{\langle \alpha, \rho_K \rangle} \times \\ &\times \prod_{\alpha \in \Delta_M^+} \frac{\langle \alpha, \lambda + \rho_K \rangle + x_1\langle \alpha, \varphi_1 \rangle + \cdots + x_m\langle \alpha, \varphi_m \rangle}{\langle \alpha, \rho_K \rangle}. \end{aligned}$$

Then $\dim F(\lambda + n_1\varphi_1 + \cdots + n_m\varphi_m) = f(n_1, \ldots, n_m)$. The degree of the polynomial f is equal to the number $|\Delta_M^+|$ of roots in Δ_M^+, and the leading term, which we denote by $\bar{f}$, is

$$\bar{f}(x_1, \ldots, x_m) = \prod_{\alpha \in \Delta_K^+ \setminus \Delta_M^+} \frac{\langle \alpha, \lambda + \rho_K \rangle}{\langle \alpha, \rho_K \rangle} \times \prod_{\alpha \in \Delta_M^+} \frac{x_1\langle \alpha, \varphi_1 \rangle + \cdots + x_m \langle \alpha, \varphi_m \rangle}{\langle \alpha, \rho_K \rangle}.$$

We define a filtered module $M_l = \oplus_{i=0}^{l} M^i$. This $\{M_l\}_l$ gives the filtration of M; and the dimension of the filtered component is

$$\dim M_l = \sum f(n_1, \ldots, n_m),$$

where the summation is over $\mathbf{n} = (n_1, \ldots, n_m) \in \mathbb{Z}_{\geqslant 0}^m$, with $n_1\varphi_1(Z) + \cdots + n_m\varphi_m(Z) \leqslant l$. We express this condition as $|\mathbf{n}| \leqslant l$ for short.

Lemma 2.2. — *Let $d = m + |\Delta_M^+|$. Then*

$$\lim_{l \to \infty} l^{-d} \sum_{|\mathbf{n}| \leqslant l} f(\mathbf{n}) = \int \bar{f}(x) dx_1 \cdots dx_m,$$

where the domain of integration is the simplex

$$\{(x_1, \ldots, x_m) \in \mathbb{R}^m \mid x_1 \geqslant 0, \ldots, x_m \geqslant 0,\ x_1\varphi_1(Z) + \cdots + x_m\varphi_m(Z) \leqslant 1\}.$$

Summarizing the above, we have the following theorem:

Theorem 2.3. — *If l is large, then*

$$\dim M_l = c \cdot l^d / d! + \text{(lower order terms)},$$

where $d = m + |\Delta_M^+|$ and

$$c = d! \prod_{\alpha \in \Delta_K^+ \setminus \Delta_M^+} \frac{\langle \alpha, \lambda + \rho_K \rangle}{\langle \alpha, \rho_K \rangle} \times \int \prod_{\alpha \in \Delta_M^+} \frac{x_1\langle \alpha, \varphi_1 \rangle + \cdots + x_m \langle \alpha, \varphi_m \rangle}{\langle \alpha, \rho_K \rangle} dx_1 \cdots dx_m,$$

with the domain of integration

$$\{(x_1, \ldots, x_m) \in \mathbb{R}^m \mid x_1 \geqslant 0, \ldots, x_m \geqslant 0,\ x_1\varphi_1(Z) + \cdots + x_m\varphi_m(Z) \leqslant 1\}.$$

2.3. Multiplicity-free action. — Let V and K be as in the Introduction. That is, in addition to the assumption made in the previous subsection, we assume that the representation V is irreducible and that $\mathbb{C}[V]$ is multiplicity-free. The set of highest weights of K-types arising in $\mathbb{C}[V]$ is a free semigroup. We denote the set of generators by $P\hat{A}^+(V)$.

Let Y be a closed irreducible K-stable subset of V. Since V has a finite number of K-orbits, Y is the closure of a K-orbit on V. We set $M = \mathbb{C}[Y]$. As in § 2.1, M can naturally be considered the quotient ring of $\mathbb{C}[V]$, and thus it inherits the natural grading from $\mathbb{C}[V]$. Then the $(\mathbb{C}[V], K)$-module M satisfies the first assumption in § 2.2, with the weight λ taken to be zero.

Lemma 2.4. — *Suppose Y is an irreducible closed K-stable subset of V. Then there exists a free semigroup $\Lambda(Y)$ such that $\mathbb{C}[Y] = \oplus_{\varphi\in\Lambda(Y)} F(\varphi)$ as a K-module. The generators of the free semigroup $\Lambda(Y)$ form a subset of $P\hat{A}^+(V)$. This subset is denoted by $P\hat{A}^+(Y) \subset P\hat{A}^+(V)$.*

A proof of this lemma is given in Ref. [6]. Also appearing there is the explicit form of the subset generating the subsemigroup, which we use in an application below.

We denote the number of elements of $P\hat{A}^+(Y)$ by m, and we set $P\hat{A}^+(Y) = \{\varphi_1, \dots, \varphi_m\}$. For a weight α, we define the vector $(\alpha_1, \dots, \alpha_m) \in \mathbb{R}^m$ by $(\langle\alpha, \varphi_1\rangle, \dots, \langle\alpha, \varphi_m\rangle)$. We define

$$\Delta_Y^+ = \{\alpha \in \Delta^+ \mid (\alpha_1, \dots, \alpha_m) \neq 0\}$$

and $k_i = \varphi_i(Z) \in \mathbb{Z}_{>0}$. Then, the K-type $F(\varphi_i)$ appears in the homogeneous component $\mathbb{C}[Y]^{k_i}$. With this notation, we can give the degree of Y.

Theorem 2.5. — *The dimension of Y is $m + |\Delta_Y^+|$, and the degree of Y is*

$$\frac{(m + |\Delta_Y^+|)!}{\prod_{\alpha\in\Delta_Y^+} \langle\alpha, \rho_K\rangle} \times \int \prod_{\alpha\in\Delta_Y^+} (\alpha_1 x_1 + \cdots + \alpha_m x_m)\, dx_1 \cdots dx_m,$$

where the domain of the integration is the simplex

$$\{(x_1, \dots, x_m) \in \mathbb{R}^m \mid x_1 \geqslant 0, \dots, x_m \geqslant 0,\ k_1 x_1 + \cdots + k_m x_m \leqslant 1\}.$$

Proof. — Applying Theorem 2.3 with

$$\bar{f}(x_1, \dots, x_m) = \prod_{\alpha\in\Delta_Y^+} \frac{\alpha_1 x_1 + \cdots + \alpha_m x_m}{\langle\alpha, \rho_K\rangle},$$

we obtain the result. □

3. Hermitian symmetric case

In this section, we consider the subclass of the multiplicity-free actions consisting of the holomorphic tangent spaces of the hermitian symmetric spaces. In this case, we can obtain a more sophisticated formula for the degree by using the structure of the restricted root system.

3.1. Hermitian Lie algebra. — We first recall some standard notation of Lie algebras, root systems and weights.

Let $\mathfrak{g}_0$ be a non-compact real simple Lie algebra. Let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be a Cartan decomposition of $\mathfrak{g}_0$. We assume that the center $\mathfrak{c}_0$ of $\mathfrak{k}_0$ is non-zero, that is, that $\mathfrak{g}_0$ is of the hermitian type. Then $\mathfrak{c}_0$ is one dimensional. Let $\mathfrak{t}_0$ be a Cartan subalgebra of $\mathfrak{k}_0$. Then $\mathfrak{t}_0$ is a compact Cartan subalgebra of $\mathfrak{g}_0$. Let $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{p}$ and $\mathfrak{t}$ denote the respective complexifications of $\mathfrak{g}_0$, $\mathfrak{k}_0$, $\mathfrak{p}_0$ and $\mathfrak{t}_0$. We denote the Killing form by $B(\cdot, \cdot)$. The restriction of the Killing form on $\mathfrak{t}$ is a non-degenerate symmetric bilinear form.