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p-ADIC BOUNDARY VALUES

by

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Abstract. — We study in detail certain natural continuous representations of $G = GL_n(K)$ in locally convex vector spaces over a locally compact, non-archimedean field K of characteristic zero. We construct boundary value maps, or integral transforms, between subquotients of the dual of a "holomorphic" representation coming from a p-adic symmetric space, and "principal series" representations constructed from locally analytic functions on G. We characterize the image of each of our integral transforms as a space of functions on G having certain transformation properties and satisfying a system of partial differential equations of hypergeometric type.

This work generalizes earlier work of Morita, who studied this type of representation of the group $SL_2(K)$. It also extends the work of Schneider-Stuhler on the De Rham cohomology of p-adic symmetric spaces. We view this work as part of a general program of developing the theory of such representations.

Introduction

In this paper, we study in detail certain natural continuous representations of $G = GL_n(K)$ in locally convex vector spaces over a locally compact, non-archimedean field K of characteristic zero. We construct boundary value maps, or integral transforms, between subquotients of the dual of a "holomorphic" representation coming from a p-adic symmetric space, and "principal series" representations constructed from locally analytic functions on G. We characterize the image of each of our integral transforms as a space of functions on G having certain transformation properties and satisfying a system of partial differential equations of hypergeometric type.

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A major motivation for studying continuous representations of p-adic groups comes from the observation that, in traditional approaches to the representation theory of p-adic groups, one separates representations into two essentially disjoint classes – the smooth representations (in the sense of Langlands theory) and the finite dimensional rational representations. Such a dichotomy does not exist for real Lie groups, where the finite dimensional representations are "smooth." The category of continuous representations which we study is broad enough to unify both smooth and rational representations, and one of the most interesting features of our results is the interaction between these two types of representations.

The principal tools of this paper are non-archimedean functional analysis, rigid geometry, and the "residue" theory developed in the paper [ST]. Indeed, the boundary value maps we study are derived from the residue map of [ST].

Before summarizing the structure of our paper and discussing our main results, we will review briefly some earlier, related results.

The pioneering work in this area is due to Morita ([Mo1-Mo6]). He intensively studied two types of representations of $SL_2(K)$. The first class of representations comes from the action of $SL_2(K)$ on sections of rigid line bundles on the one-dimensional rigid analytic space $\mathfrak{X}$ obtained by deleting the K-rational points from $\mathbf{P}^1_{/K}$; this space is often called the p-adic upper half plane. The second class of representations is constructed from locally analytic functions on $SL_2(K)$ which transform by a locally analytic character under the right action by a Borel subgroup P of $SL_2(K)$. This latter class make up what Morita called the (p-adic) principal series.

Morita showed that the duals of the "holomorphic" representations coming from the p-adic upper half plane occur as constituents of the principal series. The simplest example of this is Morita's pairing

$$(*)\qquad \Omega^1(\mathfrak{X}) \times C^{\mathrm{an}}(\mathbf{P}^1(K), K)/K \longrightarrow K$$

between the locally analytic functions on $\mathbf{P}^1(K)$ modulo constants (a "principal series" representation, obtained by induction from the trivial character) and the 1-forms on the one-dimensional symmetric space (a holomorphic representation.)

Morita's results illustrate how continuous representation theory extends the theory of smooth representations. Under the pairing (*), the locally constant functions on $\mathbf{P}^1(K)$ modulo constants (a smooth representation known as the Steinberg representation) are a G-invariant subspace which is orthogonal to the subspace of $\Omega^1(\mathfrak{X})$ consisting of exact forms. In particular, this identifies the first De Rham cohomology group of the p-adic upper half plane over K with the K-linear dual of the Steinberg representation.

The two types of representations considered by Morita (holomorphic discrete series and principal series) have been generalized to GL_n.

The "holomorphic" representations defined in [Sch] use Drinfeld's d-dimensional p-adic symmetric space $\mathfrak{X}$. The space $\mathfrak{X}$ is the complement in $\mathbf{P}^d_{/K}$ of the K-rational hyperplanes. The action of the group $G := GL_{d+1}(K)$ on $\mathbf{P}^d$ preserves the missing hyperplanes, and therefore gives an action of G on $\mathcal{X}$ and a continuous action of G on the infinite dimensional locally convex K-vector space $\mathcal{O}(\mathcal{X})$ of rigid functions on $\mathcal{X}$. The (p-adic) holomorphic discrete series representations are modelled on this example, and come from the action of G on the global sections of homogeneous vector bundles on $\mathbf{P}^d$ restricted to $\mathcal{X}$. There is a close relationship between these holomorphic representations and classical automorphic forms, coming from the theory of p-adic uniformization of Shimura varieties ([RZ], [Var]).

The second type of representation we will study are the "locally analytic" representations. Such representations are developed systematically in a recent thesis of Féaux de Lacroix ([Fea]). He defines a class of representations (which he calls "weakly analytic") in locally convex vector spaces V over K, relying on a general definition of a V-valued locally analytic function. Such a representation is a continuous linear action of G on V with the property that, for each v, the orbit maps $f_v(g) = g \cdot v$ are locally analytic V-valued functions on G. Notice that locally analytic representations include both smooth representations and rational ones.

Féaux de Lacroix's thesis develops some of the foundational properties of this type of representation. In particular, he establishes the basic properties of an induction functor (analytic coinduction). If we apply his induction to a one-dimensional locally analytic representation of a Borel subgroup of G, we obtain the p-adic principal series.

In this paper, we focus on one holomorphic representation and analyze it in terms of locally analytic principal series representations. Specifically, we study the representation of $G = GL_{d+1}(K)$ on the space $\Omega^d(\mathcal{X})$ of d-forms on the d-dimensional symmetric space $\mathcal{X}$. Our results generalize Morita, because we work in arbitrary dimensions, and Schneider-Stuhler, because we analyze all of $\Omega^d(\mathcal{X})$, not just its cohomology. Despite our narrow focus, we uncover new phenomena not apparent in either of the other works, and we believe that our results are representative of the general structure of holomorphic discrete series representations.

Our main results describe a d-step, G-invariant filtration on $\Omega^d(\mathcal{X})$ and a corresponding filtration on its continuous linear dual $\Omega^d(\mathcal{X})'$. We establish topological isomorphisms between the $d+1$ subquotients of the dual filtration and subquotients of members of the principal series. The j-th such isomorphism is given by a "boundary value map" $I^{[j]}$.

The filtration on $\Omega^d(\mathcal{X})$ comes from geometry and reflects the fact that $\mathcal{X}$ is a hyperplane complement. The first proper subspace $\Omega^d(\mathcal{X})^1$ in the filtration on $\Omega^d(\mathcal{X})$ is the space of exact forms, and the first subquotient is the d-th De Rham cohomology group.

The principal series representation which occurs as the j-th subquotient of the dual of $\Omega^d(\mathcal{X})$ is a hybrid object blending rational representations, smooth representations, and differential equations. The construction of these principal series representations is a three step process. For each $j = 0, \ldots, d$, we first construct a representation V_j of the maximal parabolic subgroup $P_{\underline{j}}$ of G having a Levi subgroup of shape $GL_j(K) \times GL_{d+1-j}(K)$. The representation V_j (which factors through this Levi subgroup) is the tensor product of a simple rational representation with the Steinberg representation of one of the Levi factors. In the second step, we apply analytic coinduction to V_j to obtain a representation of G.

The third step is probably the most striking new aspect of our work. For each j, we describe a pairing between a generalized Verma module and the representation induced from V_j. We describe a submodule $\mathfrak{d}_{\underline{j}}$ of this Verma module such that $I^{[j]}$ is a topological isomorphism onto the subspace of the induced representation annihilated by $\mathfrak{d}_{\underline{j}}$:

$$I^{[j]} : [\Omega^d(\mathfrak{X})^j / \Omega^d(\mathfrak{X})^{j+1}]' \xrightarrow{\sim} C^{\mathrm{an}}(G, P_{\underline{j}}; V_j)^{\mathfrak{d}_{\underline{j}}=0}$$

The generators of the submodules $\mathfrak{d}_{\underline{j}}$ make up a system of partial differential equations. Interestingly, these differential equations are hypergeometric equations of the type studied by Gelfand and his collaborators (see [GKZ] for example). Specifically, the equations which arise here come from the adjoint action of the maximal torus of G on the (transpose of) the unipotent radical of $P_{\underline{j}}$.

For the sake of comparison with earlier work, consider the two extreme cases when $j = 0$ and $j = d$. When $j = 0$, the group $P_{\underline{j}}$ is all of G, the representation V_j is the Steinberg representation of G, and the induction is trivial. The submodule $\mathfrak{d}_{\underline{0}}$ is the augmentation ideal of $U(\mathfrak{g})$, which automatically kills V_j because Steinberg is a smooth representation.

When $j = d$, V_d is an one-dimensional rational representation of $P_{\underline{d}}$, and the module $\mathfrak{d}_{\underline{d}}$ is zero, so that there are no differential equations. In this case we obtain an isomorphism between the bottom step in the filtration and the locally analytic sections of an explicit homogeneous line bundle on the projective space $G/P_{\underline{d}}$. When $d = 1$, these two special cases ($j = 0$ and $j = 1$) together for $SL_2(K)$ are equivalent to Morita's theory applied to $\Omega^1(\mathfrak{X})$.

We conclude this introduction with an outline of the sections of this paper. In sections one and two, we establish fundamental properties of $\Omega^d(\mathfrak{X})$ as a topological vector space and as a G-representation. For example, we show that $\Omega^d(\mathfrak{X})$ is a reflexive Fréchet space.

We introduce our first integral transform in section 2. Let ξ be the logarithmic d-form on $\mathbf{P}^d$ with first order poles along the coordinate hyperplanes. We study the map

$$\begin{array}{rcl} I : \Omega^d(\mathfrak{X})' & \longrightarrow & C^{\mathrm{an}}(G, K) \\ \lambda & \longmapsto & [g \mapsto \lambda(g_*\xi)]. \end{array}$$

We show that functions in the image of I satisfy both discrete relations and differential equations, although we are unable to precisely characterize the image of the map I.