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TORSION ÉTALE AND CRYSTALLINE COHOMOLOGIES

by

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"This cohomology should also, most importantly, explain torsion phenomena, and in particular p-torsion"
A. Grothendieck, Crystals and the de Rham cohomology of schemes.

Abstract. — Following our two courses at the Centre Émile Borel of the I.H.P. during the Semestre p-adique of 1997, we present a survey of the Fontaine-Laffaille and Fontaine-Messing theories and (with more details) of their extension by one of us to the semi-stable setting. We also very quickly discuss some ℓ-adic analogues of Nakayama. We take advantage to include a few proofs which are not in the literature and raise several remaining open questions.

1. Introduction

This article is both a resume of our two courses at the Centre Émile Borel of the I.H.P. during the Semestre p-adique and a survey of the papers [**31**], [**32**], [**9**], [**10**]. These courses were, from the outset, coordinated. Indeed, the course of the second author was largely foundational and was viewed as preparatory for the course of the first author, a Cours Peccot, devoted to his generalization to the semi-stable situation, via log-syntomic methods, of some of the results of [**32**]. We concentrate here primarily on [**9**], [**10**], adopting a strictly utilitarian point of view and, hopefully, making then the article more useful to number theorists or algebraic geometers who are not specialists in p-adic theories. Nevertheless, to keep the text to a reasonable length, we have found it necessary to assume the reader has some awareness of crystalline and semi-stable p-adic Galois representations and the corresponding comparison theorems. Certainly, an acquaintance with log-schemes would also be helpful, although we recall their definition.

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We do not intend to review, even in the most cursory fashion, the history of what has become a somewhat intricate and still evolving complex of theories and techniques all ultimately intended to clarify the relationships between the diverse p-adic objects which are cohomologically associated to appropriate algebraic varieties. These objects are either the cohomology groups or are the "coefficients" which serve as input for or arise as the output from such cohomology groups. We refer the reader to [**30**], [**25**] for discussion of the comparison conjectures and to [**41**], [**71**] for surveys of the comparison theorems in the $\mathbf{Q}_p$-coefficient context. The proofs, with varying degree of detail, are given in [**18**], [**19**], [**22**], [**32**], [**44**], [**46**], [**59**], [**70**].

The case of torsion coefficients has had itself a long gestation. The dictionary relating unramified representations and "unit root F-crystals" goes back to Artin, Hasse and (especially) Witt during the thirties. The extension of classical Dieudonné theory from the case of smooth (commutative) formal groups to finite connected or unipotent group schemes over a perfect field k is due to Gabriel ([**67**]). The analogous results over $W(k)$ are due to Fontaine ([**29**]). Grothendieck stressed both in [**34**] and in [**35**] the geometric importance of understanding p-torsion phenomena in the Picard scheme and also in higher cohomological contexts. Important examples and results were given by Mumford and Raynaud ([**55**], [**56**], [**62**]). To the best of our knowledge it was Grothendieck who, in his Algerian letter to Deligne ([**36**]), first explicitly raised the question of understanding the relation between the torsion invariants in the p-adic étale cohomology (or equivalently the Betti cohomology) of the geometric generic fiber and in the "p-adic cohomology" of the special fiber. Shortly after with the creation of crystalline cohomology ([**35**], [**3**]), it was possible to attach precise meaning to this last term. In fact, the situation is subtle as examples, due to Ekedahl ([**17**]), show that for V a complete discrete valuation ring of unequal characteristic and residue field k and X/V proper and smooth, the π-torsion invariants for $H^*_{\mathrm{dR}}(X/V)$ are not necessarily those of $H^*_{\mathrm{cris}}(X_k/W)\otimes_W V$ (where $W=W(k)$). Even today there remains much to understand concerning torsion in the (very) ramified case.

The approach we discuss in the text for studying torsion phenomena is via the use of log-syntomic methods (section 6). Although he made no application of it, it was Mazur who first discussed the syntomic topology ([**51**]). Fontaine and the second author showed in 1982 that $\mathcal{O}_n^{\mathrm{cris}}$ is a sheaf for the syntomic topology and subsequently made systematic use of syntomic methods to establish the crystalline conjecture for $e=1$ and in degree $<p$. Using Kato's K-theoretic calculations of the nearby cycles they established the equality of the torsion invariants in the same context (see section 3). It is the extension of these results to the semi-stable situation and the log-syntomic generalization of these methods which is the subject of this survey.

In the semi-stable situation, even when working over $K_0=\mathrm{Frac}(W)$, it is useful to introduce the larger ring $S_{K_0}=S\otimes_W K_0$ where $S=\widehat{W\langle u\rangle}$ is the p-adic completion of the divided power polynomial ring in the variable u. The (ϕ,N)-filtered modules D of the semi-stable theory do not satisfy Griffiths' transversality, but it is shown in [**8**] that $D \mapsto D \otimes_{K_0} S_{K_0}$ establishes an equivalence with a category of S_{K_0}-modules (equipped with additional structure) whose objects now do satisfy Griffiths' transversality (see section 4). It is the torsion analogue of this last category which generalizes in the semi-stable context the filtered module category of Fontaine and Laffaille ([**31**]). This is discussed in detail in the text. Suffice it here to say that for each r with $0 \leqslant r \leqslant p-2$, we define such a category, $\underline{\mathcal{M}}^r$ (see section 5).

The categories $\underline{\mathcal{M}}^r$ are interesting for two reasons. The first reason is that they allow one to get a handle on new and interesting phenomena in the semi-stable situation which don't arise in the analogous crystalline situation. For instance, irreducible 2-dimensional crystalline representations of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ with distinct Hodge-Tate weights in $\{0,\dots,p-2\}$ are all irreducible modulo p whereas this is far from being the case with irreducible 2-dimensional semi-stable representations of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ with distinct Hodge-Tate weights in $\{0,\dots,p-2\}$ (and here the reduction modulo p is very interesting to study, see [**9**] and the last section). The second reason is that these categories are related to geometry. Let X/W be proper and semi-stable. Endow it with its canonical log-structure (cf. section 2), denote by X_n its reduction modulo p^n and consider the log-crystalline cohomology of X_n relative to the base $E_n = \mathrm{Spec}(S/p^nS)$. This is also the log-syntomic cohomology of X with coefficients in the sheaf $\mathcal{O}_n^{\mathrm{st}}$ (which plays here the role of the classical $\mathcal{O}_n^{\mathrm{cris}}$). Then one proves that, for $0 \leqslant i \leqslant r \leqslant p-2$, the corresponding H^i (equipped with its Fil^r, ϕ_r, N) is an object of the category $\underline{\mathcal{M}}^r$ (see section 7) and, using Hyodo-Kato-Tsuji's K-theoretic calculations of the nearby cycles in the semi-stable situation, that the torsion Galois representation associated to it by the generalized Fontaine-Laffaille theory is the étale cohomology of the geometric generic fiber $X_{\overline{K}}$ with coefficients in $\mathbf{Z}/p^n\mathbf{Z}$ (see section 8).

We discuss applications of these results and related open questions in the last section. In particular we explain how to recover in the above situation the torsion invariants of the étale cohomology of the geometric generic fiber.

The reader will note that we frequently refer to the literature for the proofs. However we give proofs, or at least sketches of proofs, when a result does not have an otherwise published proof (as for instance in section 6) or when we think that the proof gives insight into the result discussed or into the techniques we use.

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2. The ℓ-torsion case

We set up the notations which we will keep throughout: p is a prime, k a perfect field of characteristic p, W the Witt vectors $W(k)$, $K_0 = \mathrm{Frac}(W)$, K a finite totally ramified extension of K_0, $\mathcal{O}_K$ its ring of integers, $\overline{K}$ an algebraic closure of K, $\mathcal{O}_{\overline{K}}$ its ring of integers, $\overline{k}$ the corresponding algebraic closure of k, and $G_K \subset G_{K_0}$ the Galois groups $\mathrm{Gal}(\overline{K}/K) \subset \mathrm{Gal}(\overline{K}/K_0)$. For any prime ℓ, recall that an ℓ-adic representation of G_K or G_{K_0} is a continuous linear representation in a finite dimensional $\mathbf{Q}_\ell$-vector space and that a (finite) ℓ-torsion representation is a continuous (and hence finite) representation of G_K or G_{K_0} in a finite length $\mathbf{Z}_\ell$-module.

2.1. Good reduction. — Let $\ell \neq p$ be another prime. As is well known, an ℓ-adic or ℓ-torsion representation of G_K that has "good reduction" is just an unramified continuous representation. One of the first and most important results of étale cohomology is certainly:

Theorem 2.1.1 **(SGA$_4$ IX.2.2 + XVI.2.2).** — *Let X be a proper smooth scheme over $\mathcal{O}_K$. For $n \in \mathbf{N}$ and $i \in \mathbf{N}$, the specialization map induces isomorphisms compatible with the action of G_K:*

$$H^i((X \times_{\mathcal{O}_K} \overline{k})_{\text{ét}}, \mathbf{Z}/\ell^n\mathbf{Z}) \xrightarrow{\sim} H^i((X \times_{\mathcal{O}_K} \overline{K})_{\text{ét}}, \mathbf{Z}/\ell^n\mathbf{Z}).$$

Notice that we compare something living on the geometric special fiber of X to something living on the geometric generic fiber. In particular, the étale cohomology of the geometric generic fiber is unramified (as a G_K-module). Till the end of this paper, we will keep this philosophy of comparing in various situations (torsion) Galois representations coming from the geometric special fiber to (torsion) Galois representations coming from the geometric generic fiber. In each case, the comparison will yield deep properties of the latter.

2.2. Semi-stable reduction. — We want to consider now the more general situation of a smooth proper K-scheme admitting a proper semi-stable model X over $\mathcal{O}_K$, that is X is regular and its special fiber is a reduced divisor with normal crossings in X. Equivalently, this means there exists an étale covering (U_i) of X such that each U_i is étale over an affine scheme of the form $\mathcal{O}_K[X_1, \ldots, X_s]/(X_1X_2\ldots X_r - \pi_K)$ $(1 \leqslant r \leqslant s)$ where π_K is an uniformizer of $\mathcal{O}_K$. We want an analogue of theorem 2.1.1 and consequently have to find a candidate to replace $H^i((X \times_{\mathcal{O}_K} \overline{k})_{\text{ét}}, \mathbf{Z}/\ell^n\mathbf{Z})$ that is still related to $X \times_{\mathcal{O}_K} k$ and that contains enough information to recover the étale cohomology of the generic fiber $X \times_{\mathcal{O}_K} K$. There is little hope the singular scheme $X \times_{\mathcal{O}_K} k$ alone will now be sufficient. What we need is some extra information related to the generic fiber, together with $X \times_{\mathcal{O}_K} k$, that is rich enough to give back the cohomology of the geometric generic fiber. It turns out that this extra information will be the log-structure (defined by Fontaine and Illusie) canonically attached to the model X (see 2.2.1.2 below). The idea is then to replace the étale cohomology of the scheme $X \times_{\mathcal{O}_K} \overline{k}$ by the *log*-étale cohomology of the *log*-scheme $X \times_{\mathcal{O}_K} \overline{k}$.