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COHOMOLOGY OF SIEGEL VARIETIES WITH p-ADIC INTEGRAL COEFFICIENTS AND APPLICATIONS

by

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Abstract. — Under the assumption that Galois representations associated to Siegel modular forms exist (it is known only for genus at most 2), we study the cohomology with p-adic integral coefficients of Siegel varieties, when localized at a non-Eisenstein maximal ideal of the Hecke algebra, provided the prime p is large with respect to the weight of the coefficient system. We show that it is torsion-free, concentrated in degree d, and that it coincides with the interior cohomology and with the intersection cohomology. The proof uses p-adic Hodge theory and the dual BGG complex modulo p in order to compute the "Hodge-Tate weights" for the mod. p cohomology. We apply this result to the construction of Hida p-adic families for symplectic groups and to the first step in the construction of a Taylor-Wiles system for these groups.

***Résumé* (Cohomologie des variétés de Siegel à coefficients entiers p-adiques et applications)**
Supposant connue l'existence des représentations galoisiennes associées aux formes modulaires de Siegel (elle ne l'est qu'en genre $\leqslant 2$ pour le moment), on étudie la cohomologie des variétés de Siegel à coefficients entiers p-adiques localisée en un idéal maximal non-Eisenstein de l'algèbre de Hecke, lorsque p est grand par rapport au poids du système de coefficients. Plus précisément, on montre qu'elle est sans torsion, concentrée en degré médian, et qu'elle coïncide avec la cohomologie d'intersection et avec la cohomologie intérieure. On utilise pour cela la théorie de Hodge p-adique et le complexe BGG dual modulo p qui calcule « les poids de Hodge-Tate » de la réduction modulo p de cette cohomologie. On applique ce résultat à la construction de familles de Hida p-ordinaires pour les groupes symplectiques et à l'ébauche de la construction d'un système de Taylor-Wiles pour ces groupes.

1. Introduction

1.1. Let G be a connected reductive group over $\mathbb{Q}$. Diamond [**16**] and Fujiwara [**29**] (independently) have axiomatized the Taylor-Wiles method which allows to study some local components $\mathbf{T}_\mathfrak{m}$ of a Hecke algebra $\mathbf{T}$ for G of suitable (minimal) level; when it applies, this method shows at the same time that $\mathbf{T}_\mathfrak{m}$ is complete intersection and that some cohomology module, viewed as a $\mathbf{T}$-module, is locally free at $\mathfrak{m}$. It

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has been successfully applied to $\mathrm{GL}(2)_{/\mathbb{Q}}$ [**73**], to some quaternionic Hilbert modular cases [**29**], and to some inner forms of unitary groups [**38**]. If one tries to treat other cases, one can let the Hecke algebra act faithfully on the middle degree Betti cohomology of an associated Shimura variety; then, one of the problems to overcome is the possible presence of torsion in the cohomology modules with p-adic integral coefficients. For $G = \mathrm{GSp}(2g)$ ($g \geqslant 1$), we want to explain in this paper why this torsion is not supported by maximal ideals of $\mathbf{T}$ which are "non-Eisenstein" and ordinary (see below for precise definitions), provided the residual characteristic p is prime to the level and greater than a natural bound. A drawback of our method is that it necessitates to assume that the existence and some local properties of the Galois representations associated to cc homological cuspidal representations on G are established. For the moment, they are proven for $g \leqslant 2$ (see below). In his recent preprint [**43**], Hida explains for the same symplectic groups G how by considering only coherent cohomology, one can let the Hecke algebra act faithfully too on cohomology modules whose torsion-freeness is built-in (without assuming any conjecture). However for some applications (like the relation, for some groups G, between special values of adjoint L-functions, congruence numbers, and cardinality of adjoint Selmer groups), the use of the Betti cohomology seems indispensable.

1.2. Let $G = \mathrm{GSp}(2g)$ be the group of symplectic similitudes given by the matrix $J = \left(\begin{smallmatrix} 0 & s \\ -s & 0 \end{smallmatrix}\right)$, whose entries are $g \times g$-matrices, and s is antidiagonal, with non-zero coefficients equal to 1; the standard Borel B, resp. torus T, in G consists in upper triangular matrices, resp. diagonal matrices in G. For any dominant weight λ for (G, B, T), we write $\widehat{\lambda}$ for its dual (that is, the dominant weight associated to the Weyl representation dual of that of λ). Let ρ be the half-sum of the positive roots. Recall that λ is given by a $(g+1)$-uple $(a_g, \ldots, a_1; c) \in \mathbb{Z}^{g+1}$ with $c \equiv a_1 + \cdots + a_g \bmod 2$, that $\widehat{\lambda} = (a_g, \ldots, a_1; -c)$ and $\rho = (g, \ldots, 1; 0)$ (see section 3.1 below). Throughout this paper, the following integer will be of great importance:

$$\boldsymbol{w} = |\lambda + \rho| = |\lambda| + d = \sum_{i=1}^{g} (a_i + i) = d + \sum_{i=1}^{g} a_i$$

where $d = g(g+1)/2$. It can be viewed as a cohomological weight as follows.

Let $\mathbb{A} = \mathbb{A}_f \times \mathbb{Q}_\infty$ be the ring of rational adèles; let G_f resp. G_∞ be the group of $\mathbb{A}_f$-points resp. $\mathbb{Q}_\infty$-points of G. Let U be a "good" open compact subgroup of $G(\mathbb{A}_f)$ (see Introd. of Sect. 2); let S resp. S_U be the Shimura variety of infinite level, resp. of level U associated to G; then $d = \dim S_U$ is the middle degree of the Betti cohomology of S_U. Let $V_\lambda(\mathbb{C})$ be the coefficient system over S resp. S_U with highest weight λ. See Sect. 2.1 for precise definitions.

Let $\pi = \pi_f \otimes \pi_\infty$ be a cuspidal automorphic representation of $G(\mathbb{A})$ which occurs in $H^d(S_U, V_\lambda(\mathbb{C}))$. This means that

– the π_f-isotypical component $W_\pi = H^d(\pi_f)$ of the G_f-module $H^\bullet(S, V_\lambda(\mathbb{C}))$ is non-zero, and

– $\pi_f^U \neq 0$.

It is known (see Sect. 2.3.1 below) that the first condition is implied by the fact that π_∞ belongs to the L-packet $\Pi_{\widehat{\lambda}+\rho}$ of Harish-Chandra's parameter $\widehat{\lambda}+\rho$ in the discrete series. In fact, it is equivalent to this fact if λ is regular or if $g = 2$.

By a Tate twist, we can restrict ourselves to the case where $c = a_g + \cdots + a_1$. We do this in the sequel. Then, $|\lambda|$ is the Deligne weight of the coefficient system V_λ and $\boldsymbol{w} = |\lambda + \rho|$ is the cohomological weight of W_π, hence the (hypothetical) motivic weight of π.

Let p be a prime. Let us fix an embedding $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. Let v be the valuation of $\overline{\mathbb{Q}}$ induced by ι_p normalized by $v(p) = 1$; let K be the v-adic completion of a number field containing the Hecke eigenvalues of π. We denote by $\mathcal{O}$ the valuation ring of (K, v); we fix a local parameter $\varpi \in \mathcal{O}$. Let N be the level of U, that is, the smallest positive integer such that the principal congruence subgroup $U(N)$ is contained in U. Let $\mathcal{H}^N$ resp. $\mathcal{H}_U(\mathcal{O})$ be the abstract Hecke algebra outside N generated over $\mathbb{Z}$, resp. over $\mathcal{O}$ by the standard Hecke operators for all primes ℓ prime to N; for any such prime ℓ, let $P_\ell(X) \in \mathcal{H}^N[X]$ be the minimal polynomial of the Hecke-Frobenius element (it is monic, of degree 2^g, see **[13]** page 247). Let $\theta_\pi : \mathcal{H}^N(\mathcal{O}) \to \mathcal{O}$ be the $\mathcal{O}$-algebra homomorphism associated to π_f.

Let $\widehat{G} = \mathrm{GSpin}_{2g+1}$ be the group of spinorial similitudes for the quadratic form

$$\sum_{i=1}^{g} 2x_i x_{2g+1-i} + x_{g+1}^2;$$

it is a split Chevalley group over $\mathbb{Z}[1/2]$ (we won't consider the prime $p = 2$ in the sequel); it can be viewed as the dual reductive group of G (see Sect. 3.2 below); let $\widehat{B}$, $\widehat{N}$, $\widehat{T}$ the standard Borel, its unipotent radical, resp. standard maximal torus therein. The group $\widehat{G}$ acts faithfully irreducibly on a space $V_{/\mathbb{Z}}$ of dimension 2^g, via the spinorial representation. Let B_V be the upper triangular Borel of GL_V. Note that $\widehat{B}$ is mapped into B_V by the spin representation.

1.3. We put $\Gamma = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. We assume that

(Gal) there exists a continuous homomorphism

$$\rho_\pi : \Gamma \longrightarrow \mathrm{GL}_V(\mathcal{O})$$

associated to π: that is, unramified outside Np, and such that the characteristic polynomial of the Frobenius element at a prime q not dividing Np is equal to $\theta_\pi(P_q(X))$.

We shall make below an assumption on the reduction of ρ_π modulo the maximal ideal of $\mathcal{O}$ which will imply that ρ_π act absolutely irreducibly on V for each geometric fiber; hence the choice of a stable $\mathcal{O}$-lattice $V_\mathcal{O}$ in $V \otimes K$ is unique up to homothety.

Evidences for **(Gal)**. — For $g = 2$, assuming

(Hol) π_∞ is in the holomorphic discrete series,

Weissauer [**87**] (see also [**34**] and [**52**]) has shown the existence of a four-dimensional p-adic Galois representation

$$\rho_\pi : \Gamma \longrightarrow \mathrm{GL}_V(\overline{\mathbb{Q}}_p)$$

Moreover, his construction, relying on trace formulae, shows actually that

$$L(W_\pi, s)^4 = L(\rho_\pi, s)^m \quad \text{for some } m \geqslant 1.$$

From this relation, one sees easily that the irreducibility of $\rho_\pi \otimes \mathrm{Id}_{\overline{\mathbb{Q}}_p}$ implies that the (Galois) semisimplification of $W_{\pi,p}$ is isomorphic to $n.\rho_\pi$ $(m = 4n)$.

Another crucial assumption for us will be that p is prime to N (hence π is unramified at p). Recall that under this assumption, Faltings has shown (Th. 6.2 (iii) of [**13**] and Th. 5.6 of [**22**]) that for any q, the p-adic representation $H^q(S_U \otimes \overline{\mathbb{Q}}_p, V_\lambda(\overline{\mathbb{Q}}_p))$ is crystalline.

Let D_p, resp. I_p be a decomposition subgroup, resp. inertia subgroup of Γ. Via the identification $X^*(T) = X_*(\widehat{T})$, we can view any $\mu \in X^*(T)$ as a cocharacter of $\widehat{T}$, hence as a homomorphism $I_p \to \mathbb{Z}_p^\times \to \widehat{T}(\mathbb{Z}_p) \to \widehat{G}(\mathbb{Z}_p)$ where the first map is the cyclotomic character $\chi : I_p \to \mathbb{Z}_p^\times$. We denote by $\widetilde{\rho}$ the character of T whose semisimple part is that of ρ, but whose central parameter is d. it is the highest weight of an irreducible representation of G given by ρ on the derived group G'. The character $\lambda + \widetilde{\rho}$ has coordinates $(a_g + g, \ldots, a_1 + 1; \boldsymbol{w})$. Let us introduce the assumption of Galois ordinarity, denoted in the sequel **(GO)**:

1) The image $\rho_\pi(D_p)$ of the decomposition group is contained in $\widehat{G}$,

2) There exists $\widehat{g} \in \widehat{G}(\mathcal{O})$ such that

$$\rho_\pi(D_p) \subset \widehat{g} \cdot \widehat{B}(\mathcal{O}) \cdot \widehat{g}^{-1},$$

3) the restriction of the conjugate $\rho_\pi^{\widehat{g}}$ to I_p, followed by the quotient by the unipotent radical $\widehat{g} \cdot \widehat{N} \cdot \widehat{g}^{-1}$ of $\widehat{g} \cdot \widehat{B} \cdot \widehat{g}^{-1}$ factors through $-(\lambda + \widetilde{\rho}) : I_p \to \widehat{T}(\mathbb{Z}_p)$.

Comments

1) Let us introduce the condition of automorphic ordinarity:

(AO) For each $r = 1, \ldots, g$,

$$v(\theta_\pi(T_{p,r})) = a_{r+1} + \cdots + a_g$$

where $T_{p,r}$ is the classical Hecke operator associated to the double class of

$$\mathrm{diag}(1_r, p \cdot 1_{2g-2r}, p^2 \cdot 1_r).$$

We conjecture that for any g, if ρ_π is residually absolutely irreducible, **(AO)** implies **(GO)**. It is well-known for $g = 1$ ([**89**] Th. 2.2.2, and [**54**]). Moreover, for $g = 2$, it follows from Proposition 7.1 of [**77**] together with a recent result of E. Urban [**80**].