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BERNSTEIN-GELFAND-GELFAND COMPLEXES AND COHOMOLOGY OF NILPOTENT GROUPS OVER $\mathbb{Z}_{(p)}$ FOR REPRESENTATIONS WITH p-SMALL WEIGHTS

by

Patrick Polo & Jacques Tilouine

Abstract. — Given a connected reductive group defined and split over $\mathbb{Z}_{(p)}$, we study Bernstein-Gelfand-Gelfand complexes over $\mathbb{Z}_{(p)}$ and prove a $\mathbb{Z}_{(p)}$-analogue of Kostant's theorem computing the $\mathfrak{n}$-homology of the Weyl module $V(\lambda)$, when λ belongs to the closure of the fundamental p-alcove.

***Résumé* (Complexes de Bernstein-Gelfand-Gelfand et cohomologie de groupes nilpotents sur $\mathbb{Z}_{(p)}$ pour les représentations de poids p-petits)**

Étant donné un groupe réductif connexe défini et déployé sur $\mathbb{Z}$, nous étudions certains complexes de Bernstein-Gelfand-Gelfand sur $\mathbb{Z}_{(p)}$ et établissons un analogue sur $\mathbb{Z}_{(p)}$ d'un théorème de Kostant, en calculant la $\mathfrak{n}$-homologie du module de Weyl $V(\lambda)$ lorsque λ appartient à l'adhérence de la p-alcôve fondamentale.

Introduction

Let G be a connected reductive linear algebraic group defined and split over $\mathbb{Z}$, let T be a maximal torus, W the Weyl group, R the root system, $R^\vee$ the set of coroots, R^+ a set of positive roots, and ρ the half-sum of the elements of R^+. Let $X = X(T)$ be the character group of T and let X^+ be the set of those $\lambda \in X$ such that $\langle \lambda, \alpha^\vee \rangle \geqslant 0$ for all $\alpha \in R^+$.

For any $\lambda \in X^+$, let $V_{\mathbb{Z}}(\lambda)$ be the Weyl module for G over $\mathbb{Z}$ with highest weight λ (see 1.3) and, for any commutative ring A, let $V_A(\lambda) = V_{\mathbb{Z}}(\lambda) \otimes_{\mathbb{Z}} A$.

Let p be a prime integer and let

$$\overline{C}_p := \{\nu \in X \mid 0 \leqslant \langle \nu + \rho, \beta^\vee \rangle \leqslant p, \quad \forall \beta \in R^+\},$$

the closure of the fundamental p-alcove.

The aim of this paper is to prove that several results about $V_{\mathbb{Q}}(\lambda)$, due to Kostant [**33**], Bernstein-Gelfand-Gelfand [**3**], Lepowsky [**37**], Rocha [**46**], and Pickel [**43**], hold

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true over $\mathbb{Z}_{(p)}$ when $\lambda \in X^+ \cap \overline{C}_p$: this is the precise meaning of the notion of p-smallness mentioned in the title.

In more details, let B be the Borel subgroup corresponding to R^+, let P be a standard parabolic subgroup containing B, let P^- be the opposed parabolic subgroup containing T, let U_P^- be its unipotent radical, and let $L = P \cap P^-$, a Levi subgroup. Let R_L be the root system of L, let $R_L^+ = R_L \cap R^+$, and

$$X_L^+ := \{\xi \in X \mid \langle \xi, \alpha^\vee \rangle \geqslant 0, \quad \forall \alpha \in R_L^+\}.$$

For any $\xi \in X_L^+$ and any commutative ring A, let $V_A^L(\xi)$ be the Weyl module for L over A with highest weight ξ.

Let $\mathfrak{g}, \mathfrak{p}, \mathfrak{u}_P^-$ be the Lie algebras over $\mathbb{Z}$ of G, P, U_P^-, respectively, and let $U(\mathfrak{g})$ and $U(\mathfrak{p})$ be the enveloping algebras of $\mathfrak{g}$ and $\mathfrak{p}$. For $\xi \in X_L^+$, consider the generalized Verma module

$$M_{\mathfrak{p}}^{\mathbb{Z}}(\xi) := U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} V_{\mathbb{Z}}^L(\xi).$$

For any commutative ring A, let $M_{\mathfrak{p}}^A(\xi) = M_{\mathfrak{p}}^{\mathbb{Z}}(\xi) \otimes_{\mathbb{Z}} A$.

Let $N = |R^+|$ and, for $i = 0, 1, \ldots, N$, let $W(i) := \{w \in W \mid \ell(w) = i\}$, where ℓ denotes the length function on W relative to B. Further, let

$$W^L = \{w \in W \mid wX^+ \subseteq X_L^+\} \quad \text{and} \quad W^L(i) := W^L \cap W(i).$$

After several recollections in Section 1, we prove in Section 2 the following Theorem (under certain restrictions on G and p, see 2.8).

Theorem A. — *Let $\lambda \in X^+ \cap \overline{C}_p$. There exists an exact sequence of $U(\mathfrak{g})$-modules:*

$$0 \longrightarrow D_N(\lambda) \longrightarrow \cdots \longrightarrow D_0(\lambda) \longrightarrow V_{\mathbb{Z}_{(p)}}(\lambda) \longrightarrow 0,$$

where each $D_i(\lambda)$ admits a finite filtration of $U(\mathfrak{g})$-submodules with associated graded

$$\operatorname{gr} D_i(\lambda) \cong \bigoplus_{w \in W^L(i)} M_{\mathfrak{p}}^{\mathbb{Z}_{(p)}}(w(\lambda + \rho) - \rho).$$

That is, following the terminology introduced in [**46**], $V_{\mathbb{Z}_{(p)}}(\lambda)$ admits a weak generalized Bernstein-Gelfand-Gelfand resolution. From this, one obtains immediately the following (see 2.1 and 2.9).

***Theorem B* (Kostant's theorem over $\mathbb{Z}_{(p)}$)**. — *Let $\lambda \in X^+ \cap \overline{C}_p$. For each i, there is an isomorphism of L-modules:*

$$H_i(\mathfrak{u}_P^-, V_{\mathbb{Z}_{(p)}}(\lambda)) \cong \bigoplus_{w \in W^L(i)} V_{\mathbb{Z}_{(p)}}^L(w(\lambda + \rho) - \rho).$$

Let $\Gamma := U_P^-(\mathbb{Z})$ be the group of $\mathbb{Z}$-points of U_P^-, it is a finitely generated, torsion free, nilpotent group. By a result of Pickel [**43**], there is a natural isomorphism $H_*(\mathfrak{u}_P^-, V_{\mathbb{Q}}(\lambda)) \cong H_*(\Gamma, V_{\mathbb{Q}}(\lambda))$. In Section 3, we prove a slightly weaker version of this result over $\mathbb{Z}_{(p)}$ when λ is p-small (see 3.8).

Theorem C. — *Let $\lambda \in X^+ \cap \overline{C}_p$. For each $n \geqslant 0$, $H_n(U_P^-(\mathbb{Z}), V_{\mathbb{Z}_{(p)}}(\lambda))$ has a natural $L(\mathbb{Z})$-module filtration such that*

$$\operatorname{gr} H_n(U_P^-(\mathbb{Z}), V_{\mathbb{Z}_{(p)}}(\lambda)) \cong \bigoplus_{w \in W^L(n)} V^L_{\mathbb{Z}_{(p)}}(w(\lambda + \rho) - \rho).$$

The proof of this result has two parts. In the first, we develop certain general results valid for any finitely generated, torsion free, nilpotent group Γ. In particular, using a beautiful theorem of Hartley [**22**], we obtain in an algebraic manner a spectral sequence relating the homology of a certain graded, torsion-free, Lie ring $\operatorname{gr}_{\mathrm{isol}} \Gamma$ associated with Γ to the homology of Γ itself, the coefficients being a Γ-module with a "nilpotent" filtration and its associated graded (see Theorem 3.5). This gives a purely algebraic, homological version (with coefficients) of a cohomological spectral sequence obtained, using methods of algebraic topology, by Cenkl and Porter [**9**]. In fact, our methods also have a cohomological counterpart. This will be developped in a subsequent paper [**44**].

In the second part of the proof, we first show that in our case where $\Gamma = U_P^-(\mathbb{Z})$, one has $\operatorname{gr}_{\mathrm{isol}} \Gamma \cong \mathfrak{u}_P^-$, and then deduce from the truth of Kostant's theorem over $\mathbb{Z}_{(p)}$ that the spectral sequence mentioned above degenerates at E_1.

Next, in Section 4, we obtain a result à la Bernstein-Gelfand-Gelfand concerning now the distribution algebras $\operatorname{Dist}(G)$ and $\operatorname{Dist}(P)$. In this case, there exists a standard *complex* (not a resolution!)

$$\mathcal{S}_\bullet(G, P, \lambda) = \operatorname{Dist}(G) \otimes_{\operatorname{Dist}(P)} (\Lambda^\bullet(\mathfrak{g}/\mathfrak{p}) \otimes V_{\mathbb{Z}}(\lambda)).$$

For $\xi \in X_L^+$, consider the generalized Verma module (for $\operatorname{Dist}(G)$ and $\operatorname{Dist}(P)$)

$$\mathcal{M}_P^{\mathbb{Z}}(\xi) := \operatorname{Dist}(G) \otimes_{\operatorname{Dist}(P)} V_{\mathbb{Z}}^L(\xi),$$

and, for any commutative ring A, set $\mathcal{S}_\bullet^A(G, P, \lambda) = \mathcal{S}_\bullet(G, P, \lambda) \otimes_{\mathbb{Z}} A$ and $\mathcal{M}_P^A(\xi) = \mathcal{M}_P^{\mathbb{Z}}(\xi) \otimes_{\mathbb{Z}} A$.

Under the assumption that $\mathfrak{u}_P^-$ is abelian, we obtain, by using an idea borrowed from [**16**, § VI.5] plus arguments from Section 2, the following result (see 4.3). Let $\mathcal{D}G$ denote the derived subgroup of G.

Theorem D. — *Assume that $\mathcal{D}G$ is simply-connected, that $X(T)/\mathbb{Z}R$ has no p-torsion and that $\mathfrak{u}_P^-$ is abelian. Let $\lambda \in X^+ \cap \overline{C}_p$. Then the standard complex $\mathcal{S}_\bullet^{\mathbb{Z}_{(p)}}(G, P, \lambda)$ contains as a direct summand a subcomplex $\mathcal{C}_\bullet^{\mathbb{Z}_{(p)}}(G, P, \lambda)$ such that, for every $i \geqslant 0$,*

$$\mathcal{C}_i^{\mathbb{Z}_{(p)}}(G, P, \lambda) \cong \bigoplus_{w \in W^L(i)} \mathcal{M}_P^{\mathbb{Z}_{(p)}}(w(\lambda + \rho) - \rho).$$

Presumably, the hypothesis that $\mathfrak{u}_P^-$ be abelian can be removed, but the proof would then require considerably more work. Since the abelian case is sufficient for the applications in the companion paper by A. Mokrane and J. Tilouine [**39**], we content ourselves with this result. We hope to come back to the general case later.

To conclude this introduction, let us mention that the results of this text are used in [**39**] in the case where G is the group of symplectic similitudes. In this case, $\mathcal{D}G$ is simply-connected and $\mathbb{Z}R$ is a direct summand of $X(T)$. When P is the Siegel parabolic, Theorem D occurs in [**39**, § 5.4] as an important step to establish a modulo p analogue of the Bernstein-Gelfand-Gelfand complex of [**16**, Chap.VI, Th. 5.5], while Theorem C (in its cohomological form) is used in [**39**, § 8.3] to study mod. p versions of Pink's theorem on higher direct images of automorphic bundles.

The notations of [**39**] follow those of [**16**] and are therefore different from the ones used in the present paper, which are standard in the theory of reductive groups. A dictionary is provided in the final section of this text.

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1. Notation and preliminaries

1.1. Let G be a connected reductive linear algebraic group, defined and split over $\mathbb{Z}$. Let T be a maximal torus, W the Weyl group, R the root system and $R^\vee$ the set of coroots. Fix a set Δ of simple roots, let R^+ and R^- be the corresponding sets of positive and negative roots, and let B, B^- denote the associated Borel subgroups and U, U^- their unipotent radicals. (For all this, see, for example, [**11**] or [**28**, § II.1]).

Let $X = X(T)$ (resp. $X^\vee = X^\vee(T)$) be the group of characters (resp. cocharacters) of T, and denote by $\langle\ ,\ \rangle$ the natural pairing between them. Elements of X will be called weights, in accordance with the terminology in Lie theory. Let $\leqslant$ denote the partial order on X defined by the positive cone $\mathbb{N}R^+$, that is, $\mu \leqslant \lambda$ if and only if $\lambda - \mu \in \mathbb{N}R^+$. Let $\mathbb{Z}R \subset X$ be the root lattice and let ρ be the half-sum of the positive roots; it belongs to $X \otimes \mathbb{Z}[1/2]$. Define, as usual, the dot action of W on X by

$$w \cdot \lambda = w(\lambda + \rho) - \rho,$$

for $\lambda \in X, w \in W$. It is easy to see that $w\rho - \rho \in \mathbb{Z}R$: applying w to the equality $2\rho = \sum_{\beta \in R^+} \beta$ and substracting, one obtains the well-known formula

$$(*) \qquad \rho - w\rho = \sum_{\beta \in R^+, w^{-1}\beta \in R^-} \beta.$$

Therefore, denoting by $N(w)$ the term on the right hand-side of $(*)$, one may also define the dot action by the formula

$$w \cdot \lambda = w\lambda - N(w),$$

from which it is clear that $w \cdot \lambda$ does indeed belong to X.

Let X^+ be the set of dominant weights:

$$X^+ := \{\lambda \in X \mid \forall \alpha \in R^+, \quad \langle \lambda, \alpha^\vee \rangle \geqslant 0\},$$

where $\alpha^\vee$ denotes the coroot associated with α.