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MICROLOCAL STUDY OF IND-SHEAVES I: MICRO-SUPPORT AND REGULARITY

by

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Abstract. — We introduce the notions of micro-support and regularity for ind-sheaves, and prove their invariance by quantized contact transformations. We apply these results to the ind-sheaves of temperate holomorphic solutions of $\mathcal{D}$-modules. We prove that the micro-support of such an ind-sheaf is the characteristic variety of the corresponding $\mathcal{D}$-module and that the ind-sheaf is regular if the $\mathcal{D}$-module is regular holonomic. We finally calculate an example of the ind-sheaf of temperate solutions of an irregular $\mathcal{D}$-module in dimension one.

Résumé **(Étude microlocale des Ind-faisceaux I: micro-support et régularité)**
Nous introduisons les notions de micro-support et régularité pour les ind-faisceaux et prouvons leur invariance par transformations de contact quantifiées. Nous appliquons ces résultats aux ind-faisceaux des solutions holomorphes tempérées des $\mathcal{D}$-modules. Nous prouvons que le micro-support d'un tel ind-faisceau est la variété caractéristique du $\mathcal{D}$-module correspondant et que le ind-faisceau est régulier si le $\mathcal{D}$-module est holonome régulier. Nous calculons enfin un exemple du ind-faisceau des solutions tempérées d'un $\mathcal{D}$-module irrégulier en dimension un.

1. Introduction

Recall that a system of linear partial differential equations on a complex manifold X is the data of a coherent module $\mathcal{M}$ over the sheaf of rings $\mathcal{D}_X$ of holomorphic differential operators. Let F be a complex of sheaves on X with $\mathbb{R}$-constructible cohomologies (one says an $\mathbb{R}$-constructible sheaf, for short). The complex of "generalized functions" associated with F is described by the complex $R\mathcal{H}om(F, \mathcal{O}_X)$, and the complex of solutions of $\mathcal{M}$ with values in this complex is described by the complex

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, R\mathcal{H}om(F, \mathcal{O}_X)).$$

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One may also microlocalize the problem by replacing $R\mathcal{H}om(F, \mathcal{O}_X)$ with $\mu hom(F, \mathcal{O}_X)$. In [4] one shows that most of the properties of this complex, especially those related to propagation or Cauchy problem, are encoded in two geometric objects, both living in the cotangent bundle T^*X, the characteristic variety of the system $\mathcal{M}$, denoted by $\mathrm{char}(\mathcal{M})$, and the micro-support of F, denoted by $SS(F)$.

The complex $R\mathcal{H}om(F, \mathcal{O}_X)$ allows us to treat various situations. For example if M is a real analytic manifold and X is a complexification of M, by taking as F the dual $D'(\mathbb{C}_M)$ of the constant sheaf on M, one obtains the sheaf $\mathcal{B}_M$ of Sato's hyperfunctions. If Z is a complex analytic hypersurface of X and $F = \mathbb{C}_Z[-1]$ is the (shifted) constant sheaf on Z, one obtains the sheaf of holomorphic functions with singularities on Z. However, the complex $R\mathcal{H}om(F, \mathcal{O}_X)$ does not allow us to treat sheaves associated with holomorphic functions with growth conditions. So far this difficulty was overcome in two cases, the temperate case including Schwartz's distributions and meromorphic functions with poles on Z and the dual case including C^∞-functions and the formal completion of $\mathcal{O}_X$ along Z. The method was two construct specific functors, the functor $T\mathcal{H}om$ of [2] and the functor $\overset{\mathrm{w}}{\otimes}$ of [5].

There is a more radical method, which consists in replacing the too narrow framework of sheaves by that of ind-sheaves, as explained in [6]. For example, the presheaf of holomorphic temperate functions on a complex manifold X (which, to a subanalytic open subset of X, associates the space of holomorphic functions with temperate growth at the boundary) is clearly not a sheaf. However it makes sense as an object (denoted by $\mathcal{O}^t_X$) of the derived category of ind-sheaves on X. Then it is natural to ask if the microlocal theory of sheaves, in particular the theory of micro-support, applies in this general setting.

In this paper we give the definition and the elementary properties of the micro-support of ind-sheaves as well as the notion of regularity.

We prove in particular that the micro-support $SS(\cdot)$ and the regular micro-support $SS_{\mathrm{reg}}(\cdot)$ of ind-sheaves behave naturally with respect to distinguished triangles and that these micro-supports are invariant by "quantized contact transformations" (in the framework of sheaf theory, as explained in [4]).

When X is a complex manifold and $\mathcal{M}$ is a coherent $\mathcal{D}_X$-module, we study the ind-sheaf $Sol^t(\mathcal{M}) := R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}^t_X)$. We prove that

(i) $SS(Sol^t(\mathcal{M})) = \mathrm{char}(\mathcal{M})$,
(ii) if $\mathcal{M}$ is regular holonomic, then $Sol^t(\mathcal{M})$ is regular.

Finally, we treat an example: we calculate the ind-sheaf of the temperate holomorphic solutions of an irregular differential equation.

This paper is the first one of a series. In Part II, we shall introduce the microlocalization functor for ind-sheaves, and in Part III we shall study the functorial behavior of micro-supports.

2. Notations and review

We will mainly follow the notations in [**4**] and [**6**].

Geometry. — In this paper, all manifolds will be real analytic (sometimes, complex analytic). Let X be a manifold. One denotes by $\tau\colon TX \to X$ the tangent bundle to X and by $\pi\colon T^*X \to X$ the cotangent bundle.

For a smooth submanifold Y of X, T_YX denotes the normal bundle to Y and T^*_YX the conormal bundle. In particular, T^*_XX is identified with X, the zero-section. For a submanifold Y of X and a subset S of X, we denote by $C_Y(S)$ the Whitney normal cone to S along Y, a conic subset of T_YX.

One denotes by $a\colon T^*X \to T^*X$ the antipodal map. If $S \subset T^*X$, one denotes by $\dot{S}$ the set $S \setminus T^*_XX$, and one denotes by S^a the image of S by the antipodal map. In particular, $\dot{T}^*X = T^*X \setminus X$, the set T^*X with the zero-section removed. One denotes by $\dot{\pi}\colon \dot{T}^*X \to X$ the projection.

If S is a locally closed subset of T^*X, we say that S is $\mathbb{R}^+$-conic (or simply "conic", for short) if it is locally invariant under the action of $\mathbb{R}^+$. If S is smooth, this is equivalent to saying that the Euler vector field on T^*X is tangent to S.

Let $f\colon X \to Y$ be a morphism of real manifolds. One has two natural maps

$$T^*X \xleftarrow[f_d]{} X \times_Y T^*Y \xrightarrow[f_\pi]{} T^*Y \tag{2.1}$$

(In [**4**], f_d is denoted by ${}^tf'$.) We denote by q_1 and q_2 the first and second projections defined on $X \times Y$.

Sheaves. — Let k be a field. We denote by $\mathrm{Mod}(k_X)$ the abelian category of sheaves of k-vector spaces and by $D^b(k_X)$ its bounded derived category.

We denote by $\mathbb{R}\text{-}\mathrm{C}(k_X)$ the abelian category of $\mathbb{R}$-constructible sheaves of k-vector spaces on X, and by $D^b_{\mathbb{R}-\mathrm{c}}(k_X)$ (resp. $D^b_{\mathrm{w}-\mathbb{R}-\mathrm{c}}(k_X)$) the full triangulated subcategory of $D^b(k_X)$ consisting of objects with $\mathbb{R}$-constructible (resp. weakly $\mathbb{R}$-constructible) cohomology. On a complex manifold, one defines similarly the categories $D^b_{\mathbb{C}-\mathrm{c}}(k_X)$ and $D^b_{\mathrm{w}-\mathbb{C}-\mathrm{c}}(k_X)$ of $\mathbb{C}$-constructible and weakly $\mathbb{C}$-constructible sheaves.

If Z is a locally closed subset of X and if F is a sheaf on X, recall that F_Z is a sheaf on X such that $F_Z|_Z \simeq F|_Z$ and $F_Z|_{X\setminus Z} \simeq 0$. One writes k_{XZ} instead of $(k_X)_Z$ and one sometimes writes k_Z instead of k_{XZ}.

If $f\colon X \to Y$ is a morphism of manifolds, one denotes by $\omega_{X/Y}$ the relative dualizing complex on X and if $Y = \{\mathrm{pt}\}$ one simply denotes it by ω_X. Recall that

$$\omega_X \simeq \mathrm{or}_X[\dim_{\mathbb{R}} X]$$

where or_X is the orientation sheaf and $\dim_R X$ is the dimension of X as a real manifold. We denote by D'_X and D_X the duality functors on $D^b(k_X)$, defined by

$$D'_X(F) = R\mathcal{H}om(F, k_X),\ D_X(F) = R\mathcal{H}om(F, \omega_X).$$

If F is an object of $D^b(k_X)$, $SS(F)$ denotes its micro-support, a closed conic involutive subset of T^*X. For an open subset U of T^*X, one denotes by $D^b(k_X;U)$ the localization of the category $D^b(k_X)$ with respect to the triangulated subcategory consisting of sheaves F such that $SS(F)\cap U=\varnothing$.

We shall also use the functor μhom as well as the operation $\widehat{+}$ and refer to loc. cit. for details.

$\mathcal{O}$-modules and $\mathcal{D}$-modules. — On a complex manifold X we consider the structural sheaf $\mathcal{O}_X$ of holomorphic functions and the sheaf $\mathcal{D}_X$ of linear holomorphic differential operators of finite order.

We denote by $\mathrm{Mod}_{\mathrm{coh}}(\mathcal{D}_X)$ the abelian category of coherent $\mathcal{D}_X$-modules. We denote by $D^b(\mathcal{D}_X)$ the bounded derived category of left $\mathcal{D}_X$-modules and by $D^b_{\mathrm{coh}}(\mathcal{D}_X)$ (resp. $D^b_{\mathrm{hol}}(\mathcal{D}_X)$), $D^b_{\mathrm{rh}}(\mathcal{D}_X)$) its full triangulated category consisting of objects with coherent cohomologies (resp. holonomic cohomologies, regular holonomic cohomologies).

Categories. — In this paper, we shall work in a given universe $\mathcal{U}$, and a category means a $\mathcal{U}$-category. If $\mathcal{C}$ is a category, $\mathcal{C}^\wedge$ denotes the category of functors from $\mathcal{C}^{\mathrm{op}}$ to **Set**. The category $\mathcal{C}^\wedge$ admits inductive limits, however, in case $\mathcal{C}$ also admits inductive limits, the Yoneda functor $h^\wedge\colon \mathcal{C}\to\mathcal{C}^\wedge$ does not commute with such limits. Hence, one denotes by $\varinjlim$ the inductive limit in $\mathcal{C}$ and by "$\varinjlim$" the inductive limit in $\mathcal{C}^\wedge$.

One denotes by $\mathrm{Ind}(\mathcal{C})$ the category of ind-objects of $\mathcal{C}$, that is the full subcategory of $\mathcal{C}^\wedge$ consisting of objects F such that there exist a small filtrant category I and a functor $\alpha\colon I\to\mathcal{C}$, with

$$F\simeq \text{“}\varinjlim\text{”}\,\alpha,\ \text{i.e., } F\simeq \text{“}\varinjlim_{i\in I}\text{”}\,F_i,\ \text{with } F_i\in\mathcal{C}.$$

The category $\mathcal{C}$ is considered as a full subcategory of $\mathrm{Ind}(\mathcal{C})$.

If $\varphi\colon\mathcal{C}\to\mathcal{C}'$ is a functor, it defines a functor $I\varphi\colon \mathrm{Ind}(\mathcal{C})\to\mathrm{Ind}(\mathcal{C}')$ which commutes with "$\varinjlim$".

If $\mathcal{C}$ is an additive category, we denote by $C(\mathcal{C})$ the category of complexes in $\mathcal{C}$ and by $K(\mathcal{C})$ the associated homotopy category. If $\mathcal{C}$ is abelian, one denotes by $D(\mathcal{C})$ its derived category. One defines as usual the full subcategories $C^*(\mathcal{C}), K^*(\mathcal{C}), D^*(\mathcal{C})$, with $*=+,-,b$. One denotes by Q the localization functor:

$$Q\colon K^*(\mathcal{C})\longrightarrow D^*(\mathcal{C}).$$

We keep the same notation Q to denote the composition $C^*(\mathcal{C})\to K^*(\mathcal{C})\to D^*(\mathcal{C})$.

Let $a,b\in\mathbb{Z}$ with $a\leqslant b$. One denotes by $C^{[a,b]}(\mathcal{C})$ the full subcategory of $C(\mathcal{C})$ consisting of objects $F^\bullet$ satisfying $F^i=0$ for $i\notin[a,b]$. There is a natural equivalence

$$\mathrm{Ind}(C^{[a,b]}(\mathcal{C}))\xrightarrow{\sim} C^{[a,b]}(\mathrm{Ind}(\mathcal{C})).$$