Yves Laurent

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REGULARITY OF $\mathcal{D}$-MODULES ASSOCIATED TO A SYMMETRIC PAIR

by

Yves Laurent

À Jean-Michel Bony, à l'occasion de son 60^e anniversaire.

Abstract. — The invariant eigendistributions on a reductive Lie algebra are solutions of a holonomic $\mathcal{D}$-module which has been proved to be regular by Kashiwara-Hotta. We solve here a conjecture of Sekiguchi saying that in the more general case of symmetric pairs, the corresponding module is still regular.

***Résumé* (Régularité des $\mathcal{D}$-modules associés à une paire symétrique).** — Sur une algèbre de Lie réductive, les distributions invariantes qui sont vecteurs propres des opérateurs différentiels bi-invariants sont les solutions d'un système holonome. Il a été démontré par Kashiwara-Hotta que ce module est régulier. Nous résolvons ici une conjecture de Sekiguchi en montrant que ce résultat est encore vrai dans le cas plus général des paires symétriques.

Introduction

Let G be a semi-simple Lie group. An irreducible representation of G has a character which is an *invariant eigendistribution*, that is a distribution on G which is invariant under the adjoint action of G and which is an eigenvalue of every biinvariant differential operator on G. A celebrated theorem of Harish-Chandra [**2**] says that all invariant eigendistributions are locally integrable functions on G.

After transfer to the Lie algebra $\mathfrak{g}$ of G by the inverse of the exponential map, an invariant eigendistribution is a solution of a $\mathcal{D}_\mathfrak{g}$-module $\mathcal{M}_\lambda^F$ for some $\lambda \in \mathfrak{g}^*$. Kashiwara and Hotta studied in [**4**] these $\mathcal{D}_\mathfrak{g}$-modules $\mathcal{M}_\lambda^F$, in particular they proved that they are holonomic and, using a modified version of the result of Harish-Chandra, proved that they are regular holonomic. This shows in particular that any hyperfunction solution of a module $\mathcal{M}_\lambda^F$ is a distribution, hence that any invariant eigenhyperfunction is a distribution.

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In [15], Sekiguchi extended the definition of the modules $\mathcal{M}_\lambda^F$ to a symmetric pair. A symmetric pair is a decomposition of a reductive Lie algebra into a direct sum of an even and an odd part, and the group associated to the even part has an action on the odd part (see section 2.1 for the details). In the diagonal case where even and odd part are identical, it is the action of a group on its Lie algebra. Sekiguchi defined a subclass of symmetric pairs ("nice pairs"), for which he proved a kind of Harish-Chandra theorem, that is that there is no hyperfunction solution of a module $\mathcal{M}_\lambda^F$ supported by a hypersurface. He also conjectured that these modules are regular holonomic.

In [11] and [12], Levasseur and Stafford give new proofs of the Harish-Chandra theorem in the original case (the "diagonal" case) and in the Sekiguchi case ("nice pairs"). In [1], we show that both theorems may be deduced from results on the roots of the b-functions associated to $\mathcal{M}_\lambda^F$.

The aim of this paper is to prove Sekiguchi's conjecture, that is the regularity of $\mathcal{M}_\lambda^F$, in the general case of symmetric pairs. Our proof do not use Harish-Chandra's theorem or its generalization, so we do not need to ask here the pairs to be "nice".

In the first section of the paper we study the regularity of holonomic $\mathcal{D}$-modules. In the definition of Kashiwara-Kawaï [6], a holonomic $\mathcal{D}$-module is regular if it is microlocally regular along each irreducible component of its characteristic variety. We had proven in [9], that the microlocal regularity may be connected to some microcharacteristic variety. We show here that an analogous result is still true if homogeneity is replaced by some quasi-homogeneity.

In the second section, we prove Sekiguchi's conjecture in theorem 2.2.1. First by standard arguments, we show that outside of the nilpotent cone, the result may be proved by reduction to a Lie algebra of lower dimension. Then on the nilpotent cone we use the results of the first section to show that the module is microlocally regular along the conormals to the nilpotent orbits.

1. Bifiltrations of $\mathcal{D}$-modules

1.1. V-filtration and microcharacteristic varieties. — In this section, we recall briefly the definitions of the V-filtration and microcharacteristic varieties. Details may be found in [10] (see also [5], [8], [13]).

Let X be a complex manifold, $\mathcal{O}_X$ be the sheaf of holomorphic functions on X and $\mathcal{D}_X$ be the sheaf of differential operators with coefficients in $\mathcal{O}_X$. Let Y be a submanifold of X. The ideal $\mathcal{I}_Y$ of holomorphic functions vanishing on Y defines a filtration of the sheaf $\mathcal{O}_X|_Y$ of functions on X defined on a neighborhood of Y by $F_Y^k\mathcal{O}_X = \mathcal{I}_Y^k$. The associate graduate, $\mathrm{gr}_Y\mathcal{O}_X = \bigoplus \mathcal{I}_Y^k/\mathcal{I}_Y^{k+1}$ is isomorphic to the sheaf $\lambda_*\mathcal{O}_{[T_YX]}$ where $\lambda : T_YX \to Y$ is the normal bundle to Y in X and $\mathcal{O}_{[T_YX]}$ the sheaf of holomorphic functions on T_YX which are polynomial in the fibers of λ. For f a function of $\mathcal{O}_X|_Y$ we will denote by $\sigma_Y(f)$ its image in $\mathrm{gr}_Y\mathcal{O}_X$.

If $\mathcal{I}$ is the ideal of definition of an analytic subvariety Z of X, then $\sigma_Y(\mathcal{I}) = \{\sigma_Y(f) \mid f \in \mathcal{I}\}$ is an ideal of $\mathcal{O}_{[T_Y X]}$ which defines the tangent cone to Z along Y [**17**].

In local coordinates (x, t) such that $Y = \{t = 0\}$, $\mathcal{I}_Y^k$ is, for $k \geqslant 0$, the sheaf of functions

$$f(x,t) = \sum_{|\alpha|=k} f_\alpha(x,t)t^\alpha$$

and if k is maximal with $f \in \mathcal{I}_Y^k$, we have $\sigma_Y(f)(x,\widetilde{t}) = \sum_{|\alpha|=k} f_\alpha(x,0)\widetilde{t}^\alpha$.

Consider now the conormal bundle to Y denoted by $\Lambda = T_Y^*X$ as a submanifold of T^*X. If f is a function on T^*X, $\sigma_\Lambda(f)$ is a function on the normal bundle $T_\Lambda(T^*X)$. The hamiltonian isomorphism $TT^*X \simeq T^*T^*X$ associated to the symplectic structure of T^*X identifies $T_\Lambda(T^*X)$ with the the cotangent bundle $T^*\Lambda$ and thus $\sigma_\Lambda(f)$ may be considered as a function on $T^*\Lambda$.

The sheaf $\mathcal{D}_X$ is provided with the filtration by the usual order of operators denoted by $(\mathcal{D}_{X,m})_{m\geqslant 0}$ and that we will call the "usual filtration". The graduate associated to this filtration is $\mathrm{gr}\mathcal{D}_X \simeq \pi_*\mathcal{O}_{[T^*X]}$ where $\pi : T^*X \to X$ is the cotangent bundle and $\mathcal{O}_{[T^*X]}$ is the sheaf of holomorphic functions polynomial in the fibers of π. We have also $\mathrm{gr}^m\mathcal{D}_X \simeq \pi_*\mathcal{O}_{[T^*X]}[m]$ where $\mathcal{O}_{[T^*X]}[m]$ is the sheaf of holomorphic functions polynomial homogeneous of degree m in the fibers of π. If P is a differential operator of $\mathcal{D}_X|_Y$, its principal symbol is a function $\sigma(P)$ on T^*X defined in a neighborhood of $\Lambda = T_Y^*X$ and $\sigma_\Lambda(\sigma(P))$ is a function on $T^*\Lambda$ (denoted by $\sigma_\Lambda\{1\}(P)$ in the notations of [**10**]).

The sheaf $\mathcal{D}_X|Y$ of differential operators on a neighborhood of Y is also provided with the the V-filtration of Kashiwara [**5**]:

$$V_k\mathcal{D}_X = \{P \in \mathcal{D}_X \mid \forall j \in \mathbb{Z},\ P\mathcal{I}_Y^j \subset \mathcal{I}_Y^{j-k}\},$$

where $\mathcal{I}_Y^j = \mathcal{O}_X$ if $j \leqslant 0$.

In local coordinates (x, t), the operators x_i and $D_{x_i} := \partial/\partial x_i$ have order 0 for the V-filtration while the operators t_i have order -1 and $D_{t_i} := \partial/\partial t_i$ order $+1$.

Remark that the V-filtration induces a filtration on $\mathrm{gr}\mathcal{D}_X \simeq \pi_*\mathcal{O}_{[T^*X]}$ which is nothing but the filtration F_Λ associated the conormal bundle $\Lambda = T_Y^*X$. In coordinates, $\Lambda = \{(x,t,\xi,\tau) \in T^*X \mid t = 0, \xi = 0\}$, a function of $\mathcal{O}_{[T^*X]}[m] \cap \mathcal{I}_\Lambda^{m-k}$ is a function $f(x,t,\xi,\tau)$ which is polynomial homogeneous of degree m in (ξ,τ) and vanishes at order at least $m-k$ on $\{t = 0,\ \xi = 0\}$.

The two filtrations of $\mathcal{D}_X$ define a bifiltration $F_{kj}\mathcal{D}_X = \mathcal{D}_{X,j} \cap V_k\mathcal{D}_X$. The associated bigraduate is defined by $\mathrm{gr}_F\mathcal{D}_X = \oplus\mathrm{gr}_F^{kj}\mathcal{D}_X$ with

$$\mathrm{gr}_F^{kj}\mathcal{D}_X = F_{kj}\mathcal{D}_X\,/(F_{k-1,j}\mathcal{D}_X + F_{k,j-1}\mathcal{D}_X)$$

and is isomorphic to $\mathrm{gr}_\Lambda \mathrm{gr}\mathcal{D}_X$ that is to the sheaf $\pi_*\mathcal{O}_{[T^*\Lambda]}$ of holomorphic functions on $T^*\Lambda$ polynomial in the fibers of $\pi : T^*\Lambda \to Y$. The image of a differential operator P in this bigraduate will be denoted by $\sigma_{\Lambda(\infty,1)}(P)$ and may be defined as follows:

If the order of P for the V-filtration is equal to the order of its principal symbol $\sigma(P)$ for the induced V-filtration then $\sigma_{\Lambda(\infty,1)}(P) = \sigma_\Lambda(\sigma(P))$ and if the order of $\sigma(P)$ is strictly lower then $\sigma_{\Lambda(\infty,1)}(P) = 0$.

Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. A good filtration of $\mathcal{M}$ is a filtration which is locally finitely generated that is locally of the form:

$$\mathcal{M}_m = \sum_{j=1,\dots,N} \mathcal{D}_{X,m+m_j} u_j,$$

where $u_1, \dots, u_N$ are (local) sections of $\mathcal{M}$ and $m_1, \dots, m_N$ integers.

It is well known that if $(\mathcal{M}_m)$ is a good filtration of $\mathcal{M}$, the associated graduate $\mathrm{gr}\mathcal{M}$ is a coherent $\mathrm{gr}\mathcal{D}_X$-module and defines the characteristic variety of $\mathcal{M}$ which is a subvariety of T^*X. This subvariety is involutive for the canonical symplectic structure of T^*X and a $\mathcal{D}_X$-module is said to be holonomic if its characteristic variety is lagrangian that is of minimal dimension.

In the same way, a good bifiltration of $\mathcal{M}$ is a bifiltration which is locally finitely generated. Then the associated bigraduate is a coherent $\mathrm{gr}_F\mathcal{D}_X$-module which defines a subvariety $\mathrm{Ch}_{\Lambda(\infty,1)}(\mathcal{M})$ of $T^*\Lambda$. It is a homogeneous involutive subvariety of $T^*\Lambda$ but it is not necessarily lagrangian even if $\mathcal{M}$ is holonomic.

If $\mathcal{I}$ is a coherent ideal of $\mathcal{D}_X$ then:

$$\mathrm{Ch}(\mathcal{M}) = \{\xi \in T^*X \mid \forall P \in \mathcal{I},\ \sigma(P)(\xi) = 0\}$$
$$\mathrm{Ch}_{\Lambda(\infty,1)}(\mathcal{M}) = \{\zeta \in T^*\Lambda \mid \forall P \in \mathcal{I},\ \sigma_{\Lambda(\infty,1)}(P)(\zeta) = 0\}$$

Regular holonomic $\mathcal{D}_X$-modules have been defined by Kashiwara and Kawaï in [**6**, Definition 1.1.16.]. A holonomic $\mathcal{D}_X$-module $\mathcal{M}$ is regular if it has regular singularities along the smooth part of each irreducible component of its characteristic variety. It is proved in [**6**] that the property of regular singularities is generic, that is it suffices to prove it on a dense open subset of Λ, in particular we may assume that Λ is the conormal bundle to a smooth subvariety of X. The definition of regular singularities along a smooth lagrangian variety is given in [**6**, Definition 1.1.11.] but in this paper, we will use the following characterization which we proved in [**9**, Theorem 3.1.7.]:

Proposition 1.1.1. — *A coherent $\mathcal{D}_X$-module has regular singularities along a lagrangian manifold Λ if and only if $\mathrm{Ch}_{\Lambda(\infty,1)}(\mathcal{M})$ is contained in the zero section of $T^*\Lambda$.*

1.2. Weighted V-filtration. — The V-filtration is associated to the Euler vector field of the normal bundle T_YX which in coordinates is equal to $\sum \widetilde{t}_i D_{\widetilde{t}_j}$. We want to