Astérisque

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Astérisque, tome 284 (2003), p. 181-244
<http://www.numdam.org/item?id=AST_2003__284__181_0>

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284, 2003, p. 181–244

BOHR-SOMMERFELD QUANTIZATION CONDITION FOR NON-SELFADJOINT OPERATORS IN DIMENSION 2

by

Anders Melin & Johannes Sjöstrand

Abstract. — For a class of non-selfadjoint h-pseudodifferential operators in dimension 2, we determine all eigenvalues in an h-independent domain in the complex plane and show that they are given by a Bohr–Sommerfeld quantization condition. No complete integrability is assumed, and as a geometrical step in our proof, we get a KAM–type theorem (without small divisors) in the complex domain.

***Résumé* (Condition de quantification de Bohr-Sommerfeld pour des opérateurs non-autoadjoints en dimension 2)**

Pour une classe d'opérateurs h-pseudodifférentiels non-autoadjoints, nous déterminons toutes les valeurs propres dans un domaine complexe indépendant de h et nous montrons que ces valeurs propres sont données par une condition de quantification de Bohr-Sommerfeld. Aucune condition d'integrabilité complète est supposée, et une étape géométrique de la démonstration est donnée par un théoreme du type KAM dans le complexe (sans petits dénominateurs).

0. Introduction

In [**MeSj**] we developed a variational approach for estimating determinants of pseudodifferential operators in the semiclassical setting, and we obtained many results and estimates of some aesthetical and philosophical value. The original purpose of the present work was to continue the study in a somewhat more special situation (see [**MeSj**], section 8) and show in that case, that our methods can lead to optimal results. This attempt turned out to be successful, but at the same time the results below are of independent interest, so the relation to the preceding work, will only be hinted upon here and there.

Let $p(x, \xi)$ be bounded and holomorphic in a tubular neighborhood of $\mathbf{R}^4$ in $\mathbf{C}^4 = \mathbf{C}^2_x \times \mathbf{C}^2_\xi$. (The assumptions near ∞ will be of importance only in the quantized case,

2000 ***Mathematics Subject Classification.*** — 31C10, 35P05, 37J40, 37K05, 47J20, 58J52.
Key words and phrases. — Bohr, Sommerfeld, eigenvalue, torus, Cauchy-Riemann equation.

and can then be be varied in many ways.) Assume that

$$\mathbf{R}^4 \cap p^{-1}(0) \neq \varnothing \text{ is connected,} \tag{0.1}$$

$$\text{on } \mathbf{R}^4 \text{ we have } |p(x,\xi)| \geqslant \frac{1}{C}, \text{ for } |(x,\xi)| \geqslant C, \tag{0.2}$$

for some $C > 0$,

(0.3) $d\,\mathrm{Re}\,p(x,\xi)$, $d\,\mathrm{Im}\,p(x,\xi)$ are linearly independent for all $(x,\xi) \in p^{-1}(0) \cap \mathbf{R}^4$.

It follows that $p^{-1}(0) \cap \mathbf{R}^4$ is a compact (2-dimensional) surface. Also assume that

$$|\{\mathrm{Re}\,p, \mathrm{Im}\,p\}| \text{ is sufficiently small on } p^{-1}(0) \cap \mathbf{R}^4. \tag{0.4}$$

Here

$$\{a, b\} = \sum_1^2 \Big(\frac{\partial a}{\partial \xi_j} \frac{\partial b}{\partial x_j} - \frac{\partial a}{\partial x_j} \frac{\partial b}{\partial \xi_j} \Big) = H_a(b)$$

is the Poisson bracket, and we adopt the following convention: We assume that p varies in some set of functions that are uniformly bounded in some fixed tube as above and satisfy (0.2), (0.3) uniformly. Then we require $|\{\mathrm{Re}\,p, \mathrm{Im}\,p\}|$ to be bounded on $p^{-1}(0) \cap \mathbf{R}^4$ by some constant > 0 which only depends on the class.

If we strengthen (0.4) to requiring that $\{\mathrm{Re}\,p, \mathrm{Im}\,p\} = 0$ on $p^{-1}(0) \cap \mathbf{R}^4$, then the latter manifold becomes Lagrangian and will carry a complex elliptic vector field $H_p = H_{\mathrm{Re}\,p} + iH_{\mathrm{Im}\,p}$. It is then a well-known topological fact (and reviewed from the point of view of analysis in appendix B of section 1) that $p^{-1}(0) \cap \mathbf{R}^4$ is (diffeomorphic to) a torus. If we only assume (0.1)–(0.4), then H_p is close to being tangent to $p^{-1}(0) \cap \mathbf{R}^4$ and the orthogonal projection of this vector field to $p^{-1}(0) \cap \mathbf{R}^4$ is still elliptic. So in this case, we have still a torus, which in general is no more Lagrangian.

In section 1 we will establish the following result:

Theorem 0.1. — *There exists a smooth 2-dimensional torus $\Gamma \subset \mathbf{C}^4$, close to $p^{-1}(0) \cap \mathbf{R}^4$ such that $\sigma_{|\Gamma} = 0$ and $I_j(\Gamma) \in \mathbf{R}$, $j = 1, 2$. Here $I_j(\Gamma) = \int_{\gamma_j} \xi \cdot dx$ are the actions along the two fundamental cycles $\gamma_1, \gamma_2 \subset \Gamma$, and $\sigma = \sum_1^2 d\xi_j \wedge dx_j$ is the complex symplectic $(2,0)$-form.*

If we form

$$L = \{\exp \widehat{tH_p}(\rho);\ \rho \in \Gamma,\ t \in \mathbf{C},\ |t| < 1/C\},$$

where $\widehat{tH_p} = tH_p + \overline{tH_p}$ is the real vector field associated to tH_p, then, as we shall see, L is a complex Lagrangian manifold $\subset p^{-1}(0)$ and L will be uniquely determined near $p^{-1}(0) \cap \mathbf{R}^4$ contrary to Γ. As a matter of fact, we will show that there is a smooth family of 2-dimensional torii $\Gamma_a \subset p^{-1}(0)$ with $\sigma_{|\Gamma_a} = 0$, depending on a complex parameter a, such that the corresponding L_a form a holomorphic foliation of $p^{-1}(0)$ near $p^{-1}(0) \cap \mathbf{R}^4$. The L_a depend holomorphically on a and so do the corresponding actions $I_j(\Gamma_a)$. We can even take one of the actions to be our complex parameter a.

It then turns out that $\operatorname{Im} \frac{dI_2}{dI_1} \neq 0$, and this implies the existence of a unique value of a for which $I_j(\Gamma_a) \in \mathbf{R}$ for $j = 1, 2$.

Theorem 0.1 can be viewed as a complex version of the KAM theorem, in a case where no small denominators are present. As pointed out to us by D. Bambusi and S. Graffi, the absence of small divisors for certain dynamical systems in the complex has been exploited by Moser [**Mo**], Bazzani–Turchetti [**BaTu**] and by Marmi–Yoccoz.

The proof we give in section 1 finally became rather simple. Using special real symplectic coordinates, we reduce the construction of the Γ_a to that of multivalued functions with single-valued gradient (from now on grad-periodic functions) on a torus, that satisfy a certain Hamilton-Jacobi equation. In suitable coordinates, this becomes a Cauchy-Riemann equation with small non-linearity and can be solved in non-integer C^m-spaces by means of a straight-forward iteration.

The fact that $I_j(\Gamma) \in \mathbf{R}$ implies that there exists an IR-manifold $\Lambda \subset \mathbf{C}^4$ (i.e. a smooth manifold for which $\sigma_{|_\Lambda}$ is real and non-degenerate) which is close to $\mathbf{R}^4$ and contains Γ. The reality of the actions $I_j(\Gamma)$ is an obvious necessary condition and the sufficiency will be established in section 1. When $p(x,\xi) \to 1$ sufficiently fast at ∞, Λ will be a critical point of the functional

$$\Lambda \longmapsto I(\Lambda) := \int_\Lambda \log|p(x,\xi)| \mu(d(x,\xi)), \tag{0.5}$$

where μ is the symplectic volume element on Λ. This was discussed in [**MeSj**] and in section 8 of that paper we also discussed the linearized problem corresponding to finding such a critical point. The reason for studying the functional (0.5) is that $I(\Lambda)$ enters in a general asymptotic upper bound on the determinant of an h-pseudodifferential operator with symbol p. Our quantum result below implies that this bound is essentially optimal.

Now let $p(x,\xi,z)$ be a uniformly bounded family of functions as above, depending holomorphically on a parameter $z \in \operatorname{neigh}(0,\mathbf{C})$ (some neighborhood of 0 in $\mathbf{C}$). Let $P(z) = p^w(x,hD,z)$ be the corresponding h-Weyl quantization of p, given by

$$p^w(x,hD,z)u(x) = \frac{1}{(2\pi h)^2} \iint e^{\frac{i}{h}(x-y)\cdot\theta} p\Big(\frac{x+y}{2},\theta,z\Big) u(y)\,dy\,d\theta. \tag{0.6}$$

It is well known (see for instance [**DiSj**]) that $P(z)$ is bounded: $L^2(\mathbf{R}^2) \to L^2(\mathbf{R}^2)$, uniformly with respect to (z,h). Moreover, the ellipticity near infinity, imposed by (0.2), implies that it is a Fredholm operator (of index 0 as will follow from the contructions below). Let us say that z is an eigen-value if $p^w(x,hD,z)$ is not bijective. The main result of our work is that the eigen-values are given by a Bohr-Sommerfeld quantization condition. We here state a shortened version (of Theorem 6.3). Let $I(z) = (I_1(z), I_2(z))$, where $I_j(z) = I_j(\Gamma(z)) \in \mathbf{R}$ and $\Gamma(z) \subset p^{-1}(0,z)$ is given by Theorem 0.1. $I(z)$ depends smoothly on z, since $\Gamma(z)$ can be chosen with smooth z-dependence.

Theorem 0.2. — *Under the above assumptions, there exists* $\theta_0 \in (\frac{1}{2}\mathbf{Z})^2$ *and* $\theta(z;h) \sim \theta_0+\theta_1(z)h+\theta_2(z)h^2+\cdots$ *in* $C^\infty(\text{neigh}\,(0,\mathbf{C});\mathbf{R}^2)$*, such that for* z *in an* h*-independent neighborhood of* 0 *and for* $h>0$ *sufficiently small, we have*
1) z *is and eigen-value iff we have*

$$\text{(BS)} \qquad \frac{I(z)}{2\pi h} = k - \theta(z;h), \ \textit{for some } k \in \mathbf{Z}^2.$$

2) When I *is a local diffeomorphism, then the eigen-values are simple (in a natural sense) and form a distorted lattice.*

Classically, the Bohr-Sommerfeld quantization condition describes the eigen-values of self-adjoint operators in dimension 1. See for instance [**HeRo**], [**GrSj**] exercise 12.3. In higher dimension Bohr-Sommerfeld conditions can still be used in the (quantum) completely integrable case for self-adjoint operators and can give all eigen-values in some interval independent of h. See for instance [**Vu**] and further references given there. This case is also intimately related to the development of Fourier integral operator theory in the version of Maslov's canonical operator theory, [**Mas**].

When dropping the integrability condition, one can still justify the BS condition and get families of eigen-values for self-adjoint operators by using quantum and classical Birkhoff normal forms, sometimes in combination with the KAM theorem, but to the authors' knowledge, no result so far describes all the eigen-values in some h-independent non-trivial interval in the self-adjoint case. See Lazutkin [**La**], Colin de Verdière [**Co**], [**Sj4**], Bambusi–Graffi–Paul [**BaGrPa**] Kaidi-Kerdelhué [**KaKe**], Popov [**Po1, Po2**]. It therefore first seems that Theorem 0.2 (6.3) is remarkable in that it describes all eigen-values in an h-independent domain and that the non-self-adjoint case (for once!) is easier to handle than the self-adjoint one. The following philosophical remark will perhaps make our result seem more natural: In dimension 1, the BS-condition gives a sequence of eigen-values that are separated by a distance $\sim h$. In higher dimension $n \geqslant 2$, this cannot hold in the self-adjoint case, since an h-independent interval will typically contain $\sim h^{-n}$ eigen-values by Weyl asymptotics, so the average separation between eigen-values is $\sim h^n$. In dimension 2 however, we can get a separation of $\sim h$ between neighboring eigen-values for non-self-adjoint operators, since the number of eigen-values in some bounded open h-independent complex domain can be bounded from above by $\mathcal{O}(h^{-2})$ by general methods.

In section 7, we study resonances of a Schrödinger operator, generated by a saddle point of the potential and apply Theorem 6.3 and its proof. In this case, the resonances in a disc of radius Ch around the corresponding critical value of the potential were determined in [**Sj2**] for every fixed $C>0$, and this result was extended by Kaidi–Kerdelhué [**KaKe**] to a description of all resonances in a disc of radius h^δ, with $\delta>0$ arbitrary but independent of h. We show that the description of [**KaKe**] extends to give all resonances in some h-independent domain.