SERGE ALINHAC

An example of blowup at infinity for a quasilinear wave equation

Astérisque, tome 284 (2003), p. 1-91

<http://www.numdam.org/item?id=AST_2003__284__1_0>

Astérisque
284, 2003, p. 1–91

AN EXAMPLE OF BLOWUP AT INFINITY FOR A QUASILINEAR WAVE EQUATION

by

Serge Alinhac

Dédié à J-M. Bony à l'occasion de son soixantième anniversaire

Abstract. — We consider an example of a Quasilinear Wave Equation which lies between the genuinely nonlinear examples (for which finite time blowup is known) and the null condition examples (for which global existence and free asymptotic behavior is known). We show global existence, though geometrical optics techniques show that the solution does not behave like a free solution at infinity. The method of proof involves commuting with fields depending on u, and uses ideas close to that of the paradifferential calculus.

***Résumé* (Explosion à l'infini pour un exemple d'équation d'ondes quasi-linéaire)**
Nous considérons un exemple d'équation d'ondes quasi-linéaire qui se situe entre les exemples vraiment non-linéaires (pour lesquels l'explosion en temps fini est connue) et les exemples vérifiant la condition nulle (pour lesquels la solution existe globalement et est asymptotiquement libre). Nous montrons l'existence globale, bien que des arguments d'optique géométrique non-linéaire indiquent un comportement non libre de la solution à l'infini. La méthode de la preuve fait intervenir la commutation avec des champs dépendant de u, et utilise des idées proches de celles du calcul paradifférentiel.

In this text, Theorems, Propositions etc. are numbered according to the section where they appear, without any mention of the Chapter. When quoted in a different chapter, they appear with the additional mention of the Chapter. For instance, in Chapter III, section 2, there is Lemma 2. In Chapter IV, section 4, the same Lemma is quoted as Lemma III.2.

2000 ***Mathematics Subject Classification.*** — 35L40.
Key words and phrases. — Quasilinear Wave Equation, Energy inequality, decay, blowup, geometrical optics, Poincaré inequality, paradifferential calculus, weighted norm.

Introduction

We prove in this paper the global existence (for ε small enough) of smooth solutions to the equation in $\mathbf{R}^3_x \times \mathbf{R}_t$

$$\partial_t^2 u - c^2(u)\Delta_x u = 0, c(u) = 1 + u,$$

with smooth and compactly supported initial data of size ε.

This result has been proved before only in the *radially symmetric case* by Lindblad [**13**], who also pointed out to some evidence that the nonradial solutions should have a very large lifespan. It turns out that the solutions do not behave at $t = +\infty$ like solutions of the free wave equation (that is, $u \sim \varepsilon/(1+t)$); most derivatives of u have, apart from the factor $\varepsilon/(1+t)$, an exponential growth $\exp C\tau$ at infinity, where $\tau = \varepsilon \log(1+t)$ is the slow time. This explains the title of this paper.

The method of proof is that of Klainerman [**11**], combining energy inequalities and commutations with appropriate "**Z**" fields. Because of the blowup at infinity, the fields we use have to be adapted to the geometry of the problem (as in Christodoulou-Klainerman [**7**]), and their coefficients smoothed out. This is very close to the paradifferential calculus of Bony [**6**], or, equivalently, to a Nash-Moser process.

I. Main result and ideas of the proof

We consider in $\mathbf{R}^3_x \times \mathbf{R}_t$ the equation

$$(1.1)_a \qquad F(u) \equiv \partial_t^2 u - c^2(u)\Delta_x u = 0,$$

where we will take for simplicity $c = c(u) = 1 + u$, since higher powers of u produce only easily handled terms. The coordinates will be

$$x = (x_1, x_2, x_3), \quad t = x_0,$$

and

$$\partial u = (\partial_1 u, \partial_2 u, \partial_3 u, \partial_t u).$$

The initial data are

$$(1.1)_b \quad u(x,0) = \varepsilon u_1^0(x) + \varepsilon^2 u_2^0(x) + \cdots, \quad (\partial_t u)(x,0) = \varepsilon u_1^1(x) + \varepsilon^2 u_2^1(x) + \cdots,$$

for real C^∞ functions u_i^j, supported in the ball $|x| \leqslant M$.

We will use the usual polar coordinates $r = |x|$, $x = r\omega$, and define the rotation fields

$$R_1 = x_2\partial_3 - x_3\partial_2, \quad R_2 = x_3\partial_1 - x_1\partial_3, \quad R_3 = x_1\partial_2 - x_2\partial_1.$$

By Z_0 we will denote one of the standard Klainerman's fields

$$(1.2) \qquad \partial_i, R_j, \quad S = t\partial_t + r\partial_r, \quad h_i = x_i\partial_t + t\partial_i.$$

For the Laplace operator, we have then

$$\Delta_x = \partial_r^2 + (2/r)\partial_r + (1/r^2)\Delta_\omega,$$

where the Laplace operator on the sphere Δ_ω is $\Delta_\omega = R_1^2 + R_2^2 + R_3^2$.

We define two linear operators

$$P \equiv c^{-1}\partial_t^2 - c\Delta, \quad P_1 \equiv c^{-1}\partial_t^2 - c(\partial_r^2 + r^{-2}\Delta_\omega), \tag{1.3}$$

such that, setting $u = \varepsilon/rU$, we have $Pu = 0$, $P_1U = 0$. We also set

$$L \equiv c^{-1/2}\partial_t + c^{1/2}\partial_r, \quad L_1 \equiv c^{-1/2}\partial_t - c^{1/2}\partial_r,$$

for which we have

$$[L, L_1] = (L_1u/2c)L_1 - (Lu/2c)L, \quad P_1 = LL_1 - cr^{-2}\Delta_\omega + (Lu/2c)L. \tag{1.4}$$

Remark that, since $c = c(u)$, iterated use of the fields $L, L_1, \partial_j, R_j, S$ will generate a considerable number of terms depending again on u. To master this phenomenon, we will have to construct an appropriate "Calculus". Finally, we set

$$\sigma_1 = M + 1 - r + t, \tag{1.5}$$

which is positive and roughly equivalent to the distance to the boundary of the light cone.

Our main result is the following Theorem.

Theorem. — *Let $s_0 \in \mathbf{N}$. For ε small enough, the Cauchy problem* (1.1) *has a global smooth solution u. Moreover, we have the estimates*

$$|Z_0^\alpha \partial u|_{L^2} \leqslant C\varepsilon(1+t)^{C\varepsilon}, \quad |\alpha| \leqslant s_0,$$

$$|\partial u| \leqslant C\varepsilon(1+t)^{-1}, \quad |Z_0^\alpha \partial u| \leqslant C\varepsilon(1+t)^{-1+C\varepsilon}\sigma_1^{-1/2}, \quad |\alpha| \leqslant s_0 - 2.$$

In the case of radially symmetric data, the solution u is a smooth function of (r^2, t). For this case, Lindblad [**13**] has proved global existence. We explain now the main ideas of the proof. In the whole paper, all constants will be denoted by C, unless otherwise specified.

I.1. A first insight using nonlinear geometrical optics

a. If w denotes the solution of the linearized problem on zero

$$(\partial_t^2 - \Delta)w = 0, \quad w(x,0) = u_1^0(x), \quad (w_t)(x,0) = u_1^1(x),$$

we know (see [**10**]) that, for some smooth F_0,

$$w \sim 1/rF_0(\omega, r-t), \quad r \to +\infty.$$

Taking εw as a rough approximation of u, we observe as in [**10**], [**1**] that the quadratic nonlinearity $u\Delta u$ produces a *slow time* effect, for the slow time $\tau \equiv \varepsilon \log(1+t)$. This means that, for large time, we expect formally u to be better approximated by

$$\varepsilon/rV(r-t, \omega, \tau),$$

for a smooth V satisfying $V(r-t,\omega,0) = F_0(\omega, r-t)$. Substituting the above expression of u in (1.1), we obtain

$$V_{\sigma\tau} + VV_{\sigma\sigma} = 0, \quad V(\sigma,\omega,0) = F_0(\omega,\sigma), \quad \sigma \equiv r-t. \tag{1.6}$$

As pointed out already in [**13**], this is in sharp contrast with what happens, for instance, for the equation $\partial_t^2 u - (1+u_t)\Delta u = 0$. In this case, a similar approach yields for V the equation $2V_{\sigma\tau} - V_\sigma V_{\sigma\sigma} = 0$, which is essentially Burgers'equation and blows up in finite time. Here, one easily sees that (1.6) has *global* solutions: this gives a hint that the lifespan of u could be very large (though not necessarily $+\infty$, see for instance the case of the null condition in two space dimensions [**1**]); the consequences of this fact are precisely stated in Theorem II.1.

b. Looking more closely, we see that the solution V of (1.6) satisfies

$$|V_\sigma| \leqslant C, \quad |\partial^\alpha_{\sigma,\omega,\tau} V| \leqslant Ce^{C\tau}.$$

Since we are willing to use Klainerman's method [**11**], we have to apply products Z_0^α to $(1.1)_a$, and use an energy inequality for P to control $|\partial Z_0^\alpha u|_{L^2}$. On the one hand, the boundedness of V_σ yields

$$|\partial u| \leqslant C\varepsilon/(1+t).$$

In the standard energy inequality for P (see [**10**] Prop. 6.3.2), this will cause an *amplification factor* of the initial energy of the form

$$\exp C\varepsilon \int_0^t ds/(1+s) = (1+t)^{C\varepsilon}.$$

Thus the best one can expect, using the energy method and Klainerman's inequality, is

$$|Z_0^\alpha \partial u| \leqslant C\varepsilon(1+t)^{-1+C\varepsilon}\sigma_1^{-1/2},$$

which is the result we obtain. On the other hand, if we believe that u and its derivatives actually behave like ε/rV, we see that derivatives like $R_i u$ or $\partial_t^2 u$, etc. do behave like $\varepsilon/r(1+t)^{C\varepsilon}$, which matches with what we just obtained from the energy method. This is why we say that we have *blowup at infinity*: the solution u exists globally, but does not behave like a solution of the linear equation. This phenomenon has been observed already, for instance in the study by Delort [**8**] of the Klein-Gordon equation.

I.2. Commuting Klainerman's fields

a. If we apply for instance a rotation field R_i to $(1.1)_a$, we obtain

$$PR_i u - 2(R_i u)(\Delta u) = 0.$$

Writing the energy inequality for P, it is not possible to reasonably absorb the term $(R_i u)(\Delta u)$ using Gronwall's lemma since

$$\exp \int_0^t |R_i u|_{L^\infty} \sim \exp[C^{-1}(1+t)^{C\varepsilon}]$$